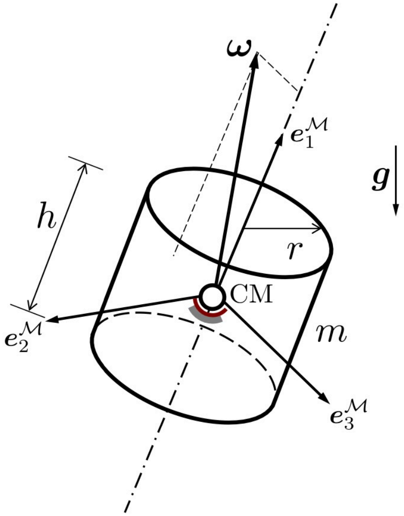
Problem 1: Rotating cylinder

A uniform cylinder of mass m, height h, radius r and principal moment of inertia tensor $\boldsymbol{I}_{\mathrm{CM}}$ is hinged at its center of mass CM. It is given an initial angular velocity $\boldsymbol{\omega}$, which lies in the $\boldsymbol{e}_1^{\mathcal{M}}$-$\boldsymbol{e}_3^{\mathcal{M}}$-plane of a principal coordinate frame rotating with the cylinder, as shown below. Gravity (acting downwards) is the only force acting on the body.

<u>Given:</u> m, ω_1, ω_3, $[\boldsymbol{I}_{\mathrm{CM}}]_{\mathcal{M}} = \begin{bmatrix} \frac{1}{2}mr^2 & 0 & 0 \\ 0 & \frac{m}{12}(3r^2+h^2) & 0 \\ 0 & 0 & \frac{m}{12}(3r^2+h^2) \end{bmatrix}$, g



What is the ratio h/r that keeps the angular velocity vector in the $\boldsymbol{e}_1^{\mathcal{M}}$-$\boldsymbol{e}_3^{\mathcal{M}}$-plane?

To solve this problem we can use the Euler equations in the principal $\mathcal{M}$-frame. As gravity acts on the CM, there are no external torques. The Euler equations then write:

i) $I_1\dot{\omega}_1 + (I_3 - I_2)\omega_3\omega_2 = 0$

ii) $I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 = 0$

iii) $I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 = 0$

We require that $\omega_2 = \dot{\omega}_2 = 0$ and we know that in general $\omega_1 \neq 0$, $\omega_3 \neq 0$. From i) and iii) we find $\dot{\omega}_1 = \dot{\omega}_3 = 0$

Now using ii) we can find a condition that $\dot{\omega}_2 = 0$:

$\Rightarrow \quad (I_1 - I_3)\,\omega_1\omega_3 = 0$ into which we can plug I_1, I_3

$$\Rightarrow \quad \left[\frac{1}{2}mr^2 - \frac{m}{12}(3r^2+h^2)\right] = 0$$

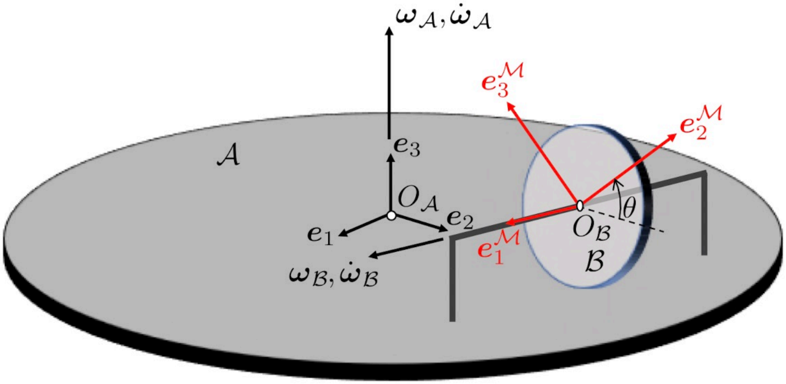
$$\frac{1}{2}r^2 = \frac{1}{4}r^2 + \frac{1}{12}h^2$$

$$3r^2 = h^2 \quad \Rightarrow \quad \underline{\underline{\frac{h}{r} = \sqrt{3}}}$$

Problem 2: Disk on a platform

A platform $\mathcal{A}$ is rotating about the fixed vertical axis $\boldsymbol{e}_3$ with a given angular velocity $\boldsymbol{\omega}_{\mathcal{A}}$ and angular acceleration $\dot{\boldsymbol{\omega}}_{\mathcal{A}}$. At the same time, an internal motor spins a uniform disk $\mathcal{B}$ about its axis of rotational symmetry (aligned with $\boldsymbol{e}_1^{\mathcal{M}}$), which is rigidly mounted to the platform, with a given angular velocity $\boldsymbol{\omega}_{\mathcal{B}}$ and angular acceleration $\dot{\boldsymbol{\omega}}_{\mathcal{B}}$. The moving frame $\{\boldsymbol{e}_1^{\mathcal{M}}, \boldsymbol{e}_2^{\mathcal{M}}, \boldsymbol{e}_3^{\mathcal{M}}\}$ is attached to $\mathcal{B}$ and centered in the center of mass $O_{\mathcal{B}}$. The frame $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is fixed and centered in the center $O_{\mathcal{A}}$ of $\mathcal{A}$. The principal moment of inertia tensor of the disk with respect to the $\mathcal{M}$-frame is given by $[\boldsymbol{I}_{O_{\mathcal{B}}}]_{\mathcal{M}}$. Neglect gravity.

<u>Given:</u> ω_A, $\dot{\omega}_A$, ω_B, $\dot{\omega}_B$, $[\boldsymbol{I}_{O_{\mathcal{B}}}]_{\mathcal{M}} = \begin{bmatrix} 2I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$, I



Determine the components $[M_i^{O_{\mathcal{B}}}]_{\mathcal{M}}$, i.e., the torque that is applied to the disk $\mathcal{B}$ at $O_{\mathcal{B}}$ by the spinning axle mounted on the rotating platform $\mathcal{A}$, measured in the moving frame $\mathcal{M}$.

We can once again use the Euler equations in the M-frame. For this we need $[\omega_i]_M$ and $[\dot{\omega}_i]_M$. We find

$$\underline{\omega} = \omega_A \underline{e}_3 + \omega_B \underline{e}_1^M$$

We need this in the M-frame, so we use geometry to find

$$\underline{e}_3 = \sin(\theta)\, \underline{e}_2^M + \cos(\theta)\, \underline{e}_3^M$$

using this we can find

$$[\underline{\omega}]_M = \begin{pmatrix} \omega_B \\ \omega_A \sin(\theta) \\ \omega_A \cos(\theta) \end{pmatrix}$$

and by differenting the components we find

$$[\dot{\underline{\omega}}]_M = \begin{pmatrix} \dot{\omega}_B \\ \dot{\omega}_A \sin(\theta) + \omega_A \omega_B \cos(\theta) \\ \dot{\omega}_A \cos(\theta) - \omega_A \omega_B \sin(\theta) \end{pmatrix}$$

where we use the chain rule and $\dot{\theta} = \omega_B$.
The Euler equations now write:

$$2I\,\dot{\omega}_B = [M_1]_M$$

$$I\left(\dot{\omega}_A \sin(\theta) + \omega_A \omega_B \cos(\theta)\right) + (2I - I)\,\omega_B \omega_A \cos(\theta) = [M_2]_M$$

$$I\left(\dot{\omega}_A \cos(\theta) - \omega_A \omega_B \sin(\theta)\right) + (I - 2I)\,\omega_B \omega_A \sin(\theta) = [M_3]_M$$

which simplifies to:

$$[M_1]_M = 2I\,\dot{\omega}_B$$

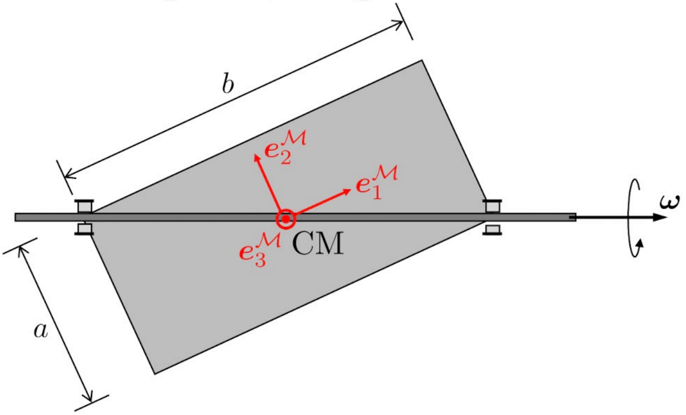
$$[M_2]_M = I\left(\dot{\omega}_A \sin(\theta) + 2\omega_B \omega_A \cos(\theta)\right)$$

$$[M_3]_M \quad I\left(\dot{\omega}_A \cos(\theta) - 2\omega_B \omega_A \sin(\theta)\right)$$

Problem 3: Rotating plate

A thin, homogeneous, rectangular plate of mass M and side lengths a and b rotates around its diagonal with a constant angular velocity $\boldsymbol{\omega}$, as indicated below. Its principal moment of inertia tensor in the given frame $\mathcal{M}$ (rotating with the plate) is denoted by $[\boldsymbol{I}_{\text{CM}}]_{\mathcal{M}}$. Neglect gravity.

Given: M, a, b, $\boldsymbol{\omega}$, $[\boldsymbol{I}_{\text{CM}}]_{\mathcal{M}} = \frac{1}{12}M \begin{bmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & (a^2+b^2) \end{bmatrix}$

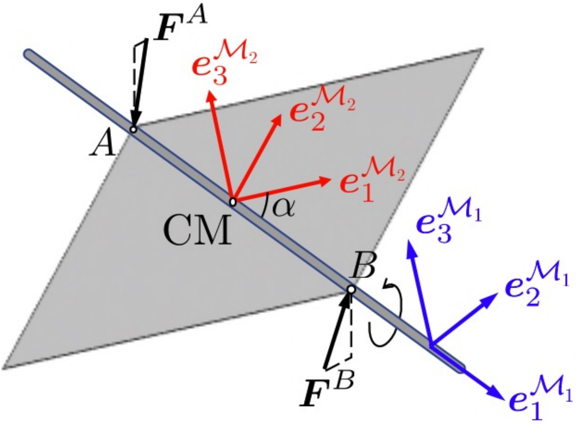


1. What is the magnitude F of the force $\boldsymbol{F}$ in the bearings?
2. What is the kinetic energy of the plate?

To solve this problem we need both AMB and LMB to get enough equations. Let us first introduce two convenient moving frames:

The frame $\mathcal{M}_1$ rotates with the shaft, while its $\underline{e}_1^{M_1}$-axis is aligned with the shaft.

The frame $\mathcal{M}_2$ rotates with the rectangle and is its principal
$\Rightarrow$ we can use the Euler equations in the $\mathcal{M}_2$-frame



In the M_1-frame the reaction forces $\underline{F}^A$, $\underline{F}^B$ are

$$\underline{F}^A = F_2^A \underline{e}_2^{M_1} + F_3^A \underline{e}_3^{M_1} \quad \text{and} \quad \underline{F}^B = F_2^B \underline{e}_2^{M_1} + F_3^B \underline{e}_3^{M_1}$$

Now let's first do LMB in the M_1-frame:

$$\underline{F}^A + \underline{F}^B = \underline{0}$$

$$\Rightarrow \left[F_2^A\right]_{M_1} = -\left[F_2^B\right]_{M_1} \quad \text{and} \quad \left[F_3^A\right]_{M_1} = -\left[F_3^B\right]_{M_1}$$

To obtain some additional equations we can use AMB, in this case the Euler equations. For this let us first find the external torques w.r.t. CM:

$$\underline{M}_{CM} = \underline{r}_{CMA} \times \underline{F}^A + \underline{r}_{CMB} \times \underline{F}^B$$

$$= -\frac{\sqrt{a^2+b^2}}{2} \underline{e}_1^{M} \times \left(F_2^A \underline{e}_2^{M_1} + F_3^A \underline{e}_3^{M_1}\right) + \frac{\sqrt{a^2+b^2}}{2} \underline{e}_1^{M} \times \left(F_2^B \underline{e}_2^{M_1} + F_3^B \underline{e}_3^{M_1}\right)$$

$$= \sqrt{a^2+b^2}\left(F_3^A \underline{e}_2^{M_1} - F_2^A \underline{e}_3^{M_1}\right)$$

The angular velocity is $\underline{\omega} = \omega \underline{e}_1^{M_1}$, with $\underline{\dot{\omega}} = \underline{0}$ (as $\omega = \text{const.}$)

Now we need to transform both $\underline{\omega}$ and $\underline{M}_{CM}$ to the M_2-frame since we can only use the Euler equations in the principal M_2-frame. For this we need the rotation tensor around the $e_3^{M_1}$-axis:

$$\left[R^{M_2 M_1}\right]_{M_2} = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Note that we do not know α, but with this we'll deal later!

Now we can transform $[\underline{\omega}]_{M_1} \to [\underline{\omega}]_{M_2}$ and $[\underline{M}_{CM}]_{M_1} \to [\underline{M}_{CM}]_{M_2}$ and plug everything into the Euler equations.

$$[\underline{\omega}]_{M_2} = \begin{pmatrix} \omega\cos(\alpha) \\ -\omega\sin(\alpha) \\ 0 \end{pmatrix}, \quad [\underline{M}_{CM}]_{M_2} = \sqrt{a^2+b^2}\begin{pmatrix} F_3^A \sin(\alpha) \\ F_3^A \cos(\alpha) \\ -F_2^A \end{pmatrix}$$

and $\underline{\dot{\omega}} = \underline{0}$

The Euler equations now write

$$0 = \sqrt{a^2+b^2}\, F_3^A \sin(\alpha)$$

$$0 = \sqrt{a^2+b^2}\, F_3^A \cos(\alpha)$$

$$-\frac{1}{12} M (b^2-a^2)\omega^2 \sin(\alpha)\cos(\alpha) = -\sqrt{a^2+b^2}\, F_2^A$$

Now we can also use the geometric definition of sin (α) and cos (α), so we don't need α :

$$\sin(\alpha) = \frac{a}{\sqrt{a^2+b^2}} \quad , \quad \cos(\alpha) = \frac{b}{\sqrt{a^2+b^2}}$$

Using this and the Euler equations we can now find

$$F_3^A = 0$$

$$F_2^A = \frac{1}{12} M\omega^2 (b^2-a^2) \frac{ab}{(a^2+b^2)^{3/2}}$$

from earlier we also know $F_2^B = -F_2^A$, $F_3^B = -F_3^A$

This means the magnitude of the reaction force is the same for both bearings, namely :

$$\underline{\underline{|F| = F_2^A}}$$

② The kinetic energy is :

$$T = \frac{1}{2}\underline{\omega}^T \underline{\underline{I}}_{CM}\underline{\omega} + \frac{1}{2} M |\cancelto{0}{\underline{v}_{CM}}|^2$$

We evaluate this in the M_2-frame as $\underline{\underline{I}}_{CM}$ is given in that frame.

$$= \frac{1}{2} I_{11} [\omega_1]^2_{M_2} + \frac{1}{2} I_{22} [\omega_2]^2_{M_2}$$

$$\underline{\underline{= \frac{1}{12} M\omega^2 \frac{a^2b^2}{a^2+b^2}}}$$