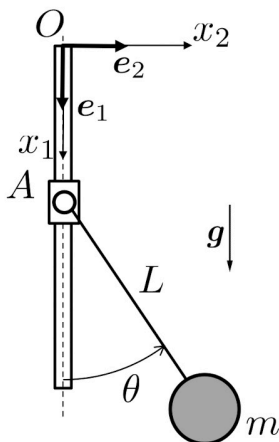


Problem 1: Particle on a slider

A particle of mass m is connected to a slider A by a massless rod of length L . The slider moves on a frictionless, vertical guide, whose position is given by $x_A(t)$. Gravity acts downwards, as shown. Denote with θ the angle that the bar encloses with the vertical axis.

Given: $m, L, g, x_A(t)$



What is the equation of motion of the system in terms of angle $\theta(t)$?

To solve this problem we first introduce a moving reference frame which rotates with the pendulum and has its origin at point A.

We can also draw a FBD:

Now we can evaluate LMB in the moving M-Frame:

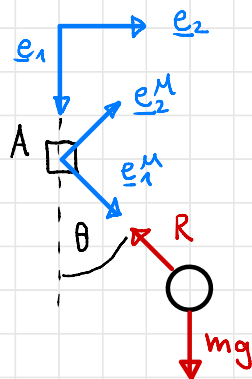
$$M \underline{a}_{cm}^M = \underline{F}_{ext} + \underline{F}_{cor} + \underline{F}_{Eu} + \underline{F}_{cent} - M \underline{a}_0^M$$

$$\text{where: } \underline{r}^M = L \underline{e}_1^M, \quad \underline{\dot{r}}^M = 0, \quad \underline{\ddot{r}}^M = \underline{a}_{cm}^M = 0$$

$$\underline{\Omega}^M = \dot{\theta} \underline{e}_3^M, \quad \underline{\dot{\Omega}} = \ddot{\theta} \underline{e}_3^M$$

using this we can find the inertial forces:

$$\underline{F}_{coriolis} = \underline{0} \quad (\text{as } \underline{v}_{cm}^M = \underline{\dot{r}}^M = 0)$$



$$\underline{F}_{\text{Euler}} = -m (\dot{\theta} \underline{e}_3^M) \times (L \underline{e}_1^M) = -m \ddot{\theta} L \underline{e}_2$$

$$\begin{aligned} \underline{F}_{\text{centrifugal}} &= -m (\dot{\theta} \underline{e}_3^M) \times (\dot{\theta} \underline{e}_3^M \times L \underline{e}_1^M) \\ &= -m (\dot{\theta} \underline{e}_3^M) \times (\dot{\theta} L \underline{e}_2^M) = m \dot{\theta}^2 L \underline{e}_1^M \end{aligned}$$

$$\underline{F}_{\text{ext}} = (-R + mg \cos(\theta)) \underline{e}_1^M - mg \sin(\theta) \underline{e}_2^M$$

and the acceleration of the origin (i.e. point A) is:

$$\underline{a}_O^M = \underline{a}_A^M = \ddot{x}_A \cos(\theta) \underline{e}_1^M - \ddot{x}_A \sin(\theta) \underline{e}_2^M$$

Plugging everything into LMB gives us:

$$\underline{e}_1^M: \quad 0 = (-R + mg \cos(\theta)) + m \dot{\theta}^2 L - m \ddot{x}_A \cos(\theta)$$

$$\underline{e}_2^M: \quad 0 = -mg \sin(\theta) - m \ddot{\theta} L + m \ddot{x}_A \sin(\theta)$$

As we are looking for the equation of motion we only need the second equation, which becomes:

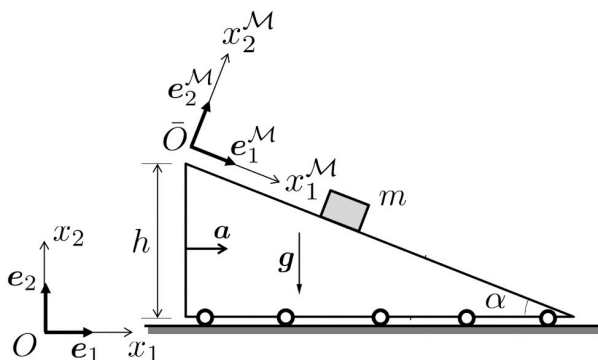
$$\underline{L \ddot{\theta} + (g - \ddot{x}_A) \sin(\theta) = 0}$$

If we also wanted to find \underline{R} , we would have to solve the equation of motion, which in this case is a hard problem.

Problem 2: Block on an incline

A block of mass m slides down a frictionless incline, which translates at a constant acceleration $\mathbf{a} = a\mathbf{e}_1$. The inclination angle is α . Gravity acts downwards, as indicated.

Given: m, h, α, a, g



1. What is the acceleration of the block relative to the moving incline?
2. What is the magnitude of the normal force between the block and incline?

1) We can once again use LMB for the moving frame. Since the frame does not rotate, all inertial forces vanish:

$$M \underline{a}_{CM}^M = \underline{F}_{ext} - M \underline{a}_O^M$$

where:

$$\underline{F}_{ext} = mg \sin(\alpha) \underline{e}_1^M - mg \cos(\alpha) \underline{e}_2^M + N \underline{e}_2^M$$

$$\underline{a}_O^M = a \cos(\alpha) \underline{e}_1^M + \alpha \sin(\alpha) \underline{e}_2^M$$

We also know that $x_2^M = 0$ at all times, as the block does not leave the plane.

We can now plug everything into LMB, using

$$\underline{a}_{CM}^M = \overset{00}{x}_1^M \underline{e}_1^M + \overset{00}{x}_2^M \underline{e}_2^M$$

which gives:

$$\underline{e}_1^M: \underline{\ddot{x}}_1^M = \underline{g \sin(\alpha) - a \cos(\alpha)}$$

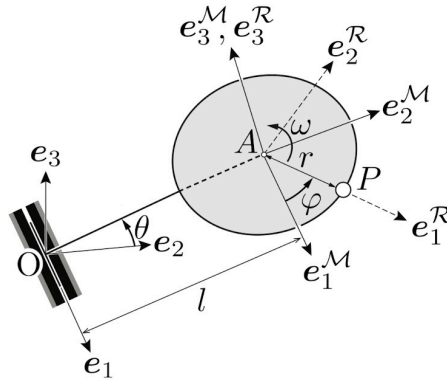
$$\underline{e}_2^M: 0 = -mg \cos(\alpha) - a \sin(\alpha) + N$$

$$\underline{N} = \underline{mg \cos(\alpha) + a \sin(\alpha)}$$

Problem 3: Spinning carousel

A carousel is turning with a time-dependent angle $\theta(t)$ around the e_1 -axis at a distance l from its center. In addition, it spins with a constant angular velocity $\omega = \omega e_3^M$ about its center of mass. A carousel seat is located at point P at a distance r from the center of the carousel. To ensure the well-being of the passengers, the (absolute) velocity and acceleration must be evaluated. Neglect gravity.

Given: l, r, ω



- (a) What are the components of the *absolute* velocity and *absolute* acceleration of P in the $\{e_1^M, e_2^M, e_3^M\}$ -frame?

To solve this problem we need two formulas from the lecture notes:

$$i) \quad \dot{\underline{r}}_P^C = \dot{\underline{r}}_{OM} + \dot{\underline{r}}_{OMP}^M + \underline{\Omega}^M \times \underline{r}_{OMP}^M$$

$$ii) \quad \ddot{\underline{r}}_P^C = \ddot{\underline{r}}_{OM} + \ddot{\underline{r}}_{OMP}^M + \underbrace{\frac{d\underline{\Omega}^M}{dt}}_{\text{Euler}} \times \underline{r}_{OMP}^M + \underbrace{2\underline{\Omega}^M \times \dot{\underline{r}}_{OMP}^M}_{\text{Coriolis}} + \underbrace{\underline{\Omega}^M \times (\underline{\Omega}^M \times \underline{r}_{OMP}^M)}_{\text{Centripetal}}$$

Where C is the fixed and M the moving frame, in our case

$$C = \{e_1, e_2, e_3\} \text{ and } M = \{e_1^M, e_2^M, e_3^M\}$$

We will also abbreviate \underline{r}_{OM} as \underline{r}_A and \underline{r}_{OMP} as \underline{r}_{AP}

Now let us find all the necessary quantities to evaluate i) and ii)

First we need the derivative of the M -basis vectors using:

$$\dot{\underline{e}}_i^M = \underline{\Omega}^M \times \underline{e}_i^M, \text{ where } \underline{\Omega}^M = \dot{\theta} \underline{e}_1^M, \quad \frac{d\underline{\Omega}^M}{dt} = \ddot{\theta} \underline{e}_1^M$$

$$\Rightarrow \dot{\underline{e}}_1^M = 0, \quad \dot{\underline{e}}_2^M = \dot{\theta} \underline{e}_3^M, \quad \dot{\underline{e}}_3^M = -\dot{\theta} \underline{e}_2^M$$

Now we can find the positions $\underline{r}_A, \underline{r}_{AP}$ and their derivatives:

$$\bullet \underline{r}_A = L \underline{e}_2^M, \quad \dot{\underline{r}}_A = L \dot{\theta} \underline{e}_3^M, \quad \ddot{\underline{r}}_A = L \ddot{\theta} \underline{e}_3^M - L \dot{\theta}^2 \underline{e}_2^M$$

$$\bullet \underline{r}_{AP} = r \cos(\varphi) \underline{e}_1^M + r \sin(\varphi) \underline{e}_2^M$$

$$\dot{\underline{r}}_{AP} = r\omega (-\sin(\varphi) \underline{e}_1^M + \cos(\varphi) \underline{e}_2^M)$$

$$\ddot{\underline{r}}_{AP} = -r\omega^2 (\cos(\varphi) \underline{e}_1^M + \sin(\varphi) \underline{e}_2^M)$$

↑ Note that these are the absolute derivatives

← and these are relative which means they are as seen from the M -frame

All of this we can now plug into i) and ii) and find:

$$\dot{\underline{r}}_p^C = -r\omega \sin(\varphi) \underline{e}_1^M + r \cos(\varphi) \underline{e}_2^M + (L + r \sin(\varphi)) \dot{\theta} \underline{e}_3^M$$

$$\ddot{\underline{r}}_p^C = -r\omega^2 \cos(\varphi) \underline{e}_1^M - [(L + r \sin(\varphi)) \dot{\theta}^2 + r\omega^2 \sin(\varphi)] \underline{e}_2^M \\ + [(L + r \sin(\varphi)) \ddot{\theta} + 2r\omega \dot{\theta} \cos(\varphi)] \underline{e}_3^M$$