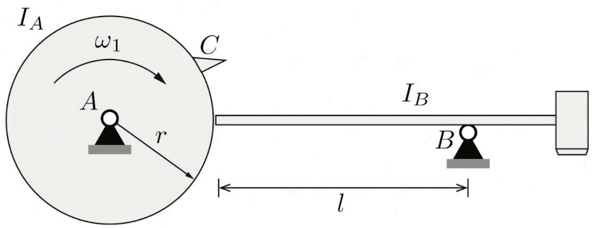
Problem 1: Hammer mechanism

A hammer mill consists of a uniform disk of radius r with moment of inertia I_A hinged at point A and a hammer with moment of inertia I_B that is hinged at a distance l from the disk. The disk rotates initially with an angular velocity ω_1 until its thumb C hits the shaft of the hammer. The coefficient of restitution of this impact is a known value e. The hammer is at rest before the impact. You may assume that the thumb C is of negligible mass and dimension. Neglect gravity.

Given: I_A, I_B, r, l, e, ω_1

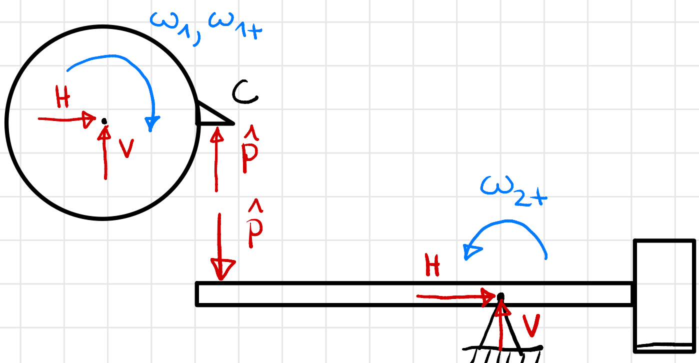


1. Determine the angular velocity of the disk and hammer immediately after the impact.

For the next task, assume that the angular velocity of the disk and hammer immediately after the impact are given by $\omega_{1+} = \frac{\omega_1}{2}$ and $\omega_{2+} = \frac{\omega_1}{2}\frac{r}{l}$, respectively, and that $I_B/I_A = l^2/r^2$.

2. Determine the dissipated (lost) energy.

To start let's draw a free-body diagram of the system:



In this case $\hat{P}$ is the impulsive force which acts during the impact

$$\underline{\hat{P}} = \int_{t_-}^{t_+} \underline{P}(t)\,dt$$

To find this $\hat{P}$ we can integrate AMB over the time of the impact. For the disk we get:

$$I_A(\omega_{1+} - \omega_{1-}) = -r\hat{P},$$

where ω_{1-} is just ω_1. If we do the same for the hammer we get

$$I_B \cdot \omega_{2+} = L\hat{P}$$

We can combine these two into one equation:

$$\frac{I_B\,\omega_{2+}}{L} = -\frac{I_A(\omega_{1+}-\omega_1)}{r} \qquad (i)$$

Now we have one equation for two variables, for a second equation we can use the coefficient of restitution:

$$e = -\frac{V_{1+}-V_{2+}}{V_{1-}-V_{2-}}$$

We can relate these velocities using kinematics:

$$V_{1+} = r\omega_{1+}, \qquad V_{2+} = L\omega_{2+}$$

$$V_{1-} = r\omega_{1-}, \qquad V_{2-} = 0 \quad \text{(start from rest)}$$

$$\Rightarrow \quad e = -\frac{r\omega_{1+} - L\omega_{2+}}{r\omega_1} \quad \Rightarrow \quad \omega_{1+} = \frac{L}{r}\omega_{2+} - e\omega_1 \qquad (ii)$$

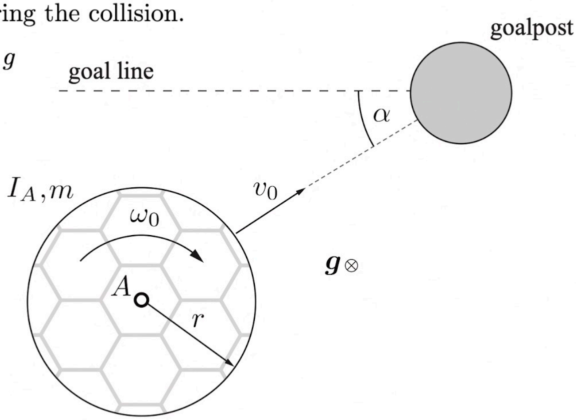
Now we can plug (ii) into (i) and solve for ω_{2+}, which we can plug into (ii) and find ω_{1+}

$$\omega_{2+} = \omega_1 \frac{(1+e)\frac{r}{L}}{1+\frac{r^2 I_B}{L^2 I_A}} \qquad \omega_{1+} = \omega_1 \frac{1 - e\frac{r^2 I_B}{L^2 I_A}}{1+\frac{r^2 I_B}{L^2 I_A}}$$

Problem 2: The decisive shot

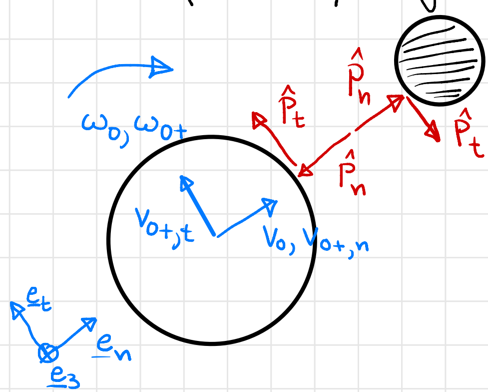
A soccer ball of mass m and moment of inertia I_A hits a *rough* goalpost *centrical* with initial velocity v_0 under an angle α to the goal line, with a known coefficient of restitution e. Assume that the goalpost is immovable and rigid. Gravity acts perpendicular to the plane, as indicated. Assume that the ball *sticks* to the rough goalpost during the collision.

Given: I_A, m, r, α, e, v_0, g



What is the minimum required initial spin (i.e., the minimum initial angular velocity ω_0) of the ball to cross the goal line (assuming that it eventually crosses the goal line if it is deflected by some angle greater than α)?

Once again we start with a free body diagram:



We can now integrate AMB and LMB over the impact:

$$\text{LMB:} \quad m\,(v_{0+,n} - v_0)\,\underline{e}_n = -\hat{P}_n\,\underline{e}_n$$

$$m \cdot v_{0+,t}\,\underline{e}_t = \hat{P}_t\,\underline{e}_t$$

$$\text{AMB:} \quad I_A(\omega_{0+} - \omega_0)\,\underline{e}_3 = -r\hat{P}_t\,\underline{e}_3$$

We can also use the coefficien of restitution to relate V_0 and $V_{0+,n}$
Note that the goalpost is rigid i.e. always has zero velocity.

$$e = -\frac{V_{0+,n}}{V_0} \Rightarrow V_{0+,n} = -eV_0$$

We also know that the ball sticks to the post. This means we can enforce rolling without slipping and relate $V_{0+,t}$ and ω_{0+}

$$V_{0+,t} = r\omega_{0+}$$

We can now plug this into our LMB and AMB and recover:

$$V_{0+,t} = \frac{r\omega_0}{1 + \frac{r^2 m}{I_A}}$$

Now we know $V_{0+,t}$ and $V_{0+,n}$ we to ensure that the ball crosses the line:

$$V_{0+,n}\sin(\alpha) + V_{0+,t}\cos(\alpha) > 0$$

We can plug the velocities we found into this and solve this to find a condition for ω_0:

$$\omega_0 > \frac{eV_0}{r}\left(1 + \frac{r^2 m}{I_A}\right)\tan(\alpha)$$

Problem 5: Active and passive rotations

Let $\boldsymbol{v} \in \mathbb{R}^3$ be a vector, whose components measured in a frame $\mathcal{C}$ with fixed Cartesian basis $\{\boldsymbol{e}_1^{\mathcal{C}}, \boldsymbol{e}_2^{\mathcal{C}}, \boldsymbol{e}_3^{\mathcal{C}}\} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$, are

$$[\boldsymbol{v}]_{\mathcal{C}} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{40}$$

Consider a second orthonormal frame $\mathcal{M}$, defined by the following base vectors:

$$[\boldsymbol{e}_1^{\mathcal{M}}]_{\mathcal{C}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad [\boldsymbol{e}_2^{\mathcal{M}}]_{\mathcal{C}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \qquad [\boldsymbol{e}_3^{\mathcal{M}}]_{\mathcal{C}} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}. \tag{41}$$

1. What are the components of $\boldsymbol{v}$ in frame $\mathcal{M}$, i.e., what is $[\boldsymbol{v}]_{\mathcal{M}}$?
2. A new vector $\boldsymbol{u}$ is obtained by rotating the given vector $\boldsymbol{v}$ by $\pi/2$ about the $\boldsymbol{e}_1^{\mathcal{M}}$-axis, then by $\pi/4$ about the $\boldsymbol{e}_2^{\mathcal{M}}$-axis, and then again by $\pi/2$ about the $\boldsymbol{e}_1^{\mathcal{M}}$-axis. What are the components of $\boldsymbol{u}$ in the $\mathcal{M}$-frame? Is $\boldsymbol{u}$ equal to $\boldsymbol{v}$?

1. To find $[v]_\mathcal{M}$, we need to project $[v]_\mathcal{C}$ onto the basis of the $\mathcal{M}$-frame.

$$[v]_\mathcal{M} = [T^{\mathcal{MC}}]_\mathcal{C} [v]_\mathcal{C}$$

where the transformation matrix consists of the basis vectors $[e_i^\mathcal{M}]_\mathcal{C}$ stacked horizontally

$$[T^{\mathcal{MC}}]_\mathcal{C} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\Rightarrow [v]_\mathcal{M} = (\sqrt{2}, 0, -1)^T$$

2. To find $\underline{u}$ we need to rotate $\underline{v}$ about all 3 axes using rotation matrices

$$[R]_\mathcal{M} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \pi/2 & -\sin \pi/2 \\ 0 & \sin \pi/2 & \cos \pi/2 \end{pmatrix}}_{\text{3rd rotation}} \cdot \underbrace{\begin{pmatrix} \cos \pi/4 & 0 & \sin \pi/4 \\ 0 & 1 & 0 \\ -\sin \pi/4 & 0 & \cos \pi/4 \end{pmatrix}}_{\text{2nd rot.}} \cdot \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \pi/2 & -\sin \pi/2 \\ 0 & \sin \pi/2 & \cos \pi/2 \end{pmatrix}}_{\text{1st rot.}}$$

Note the order of the rotations: we always multiply the next rotation on the left.

$$\Rightarrow [R]_M = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\Rightarrow [u]_M = [R]_M [v]_M = (1, 1, 1)^T$$

We note that $[v]_C = [u]_M$ does not mean that $u = v$, as only their components in two different frames are the same.

$[v]_M \neq [u]_M$!