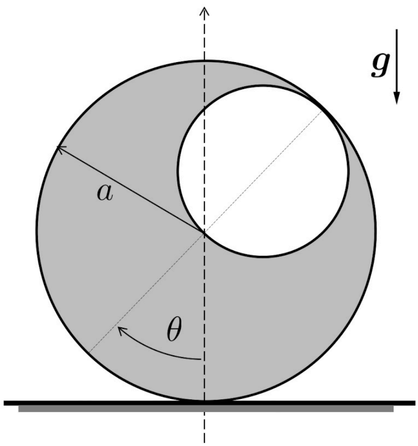
Problem 1: Cylinder with a hole

A rigid circular cylinder of radius a and thickness h has an offset hole of radius $a/2$, as shown below. The density ρ of the cylinder is uniform. Assume that the cylinder rolls without slipping on the floor. Gravity acts downwards, as indicated.

Given: a, ρ, h, g



What are the kinetic energy T and the potential energy V of the cylinder as functions of angle θ?

Hint: It may be helpful to use that $c_1^2 = c_2^2 + c_3^2 - 2c_2c_3\cos\gamma$ holds for an arbitrary triangle with sides c_1, c_2, c_3, where γ is the angle opposite of c_1 (law of cosines).

The kinetic energy is generally:

$$T = \frac{1}{2} M v_C^2 + \frac{1}{2} I_C \omega^2,$$

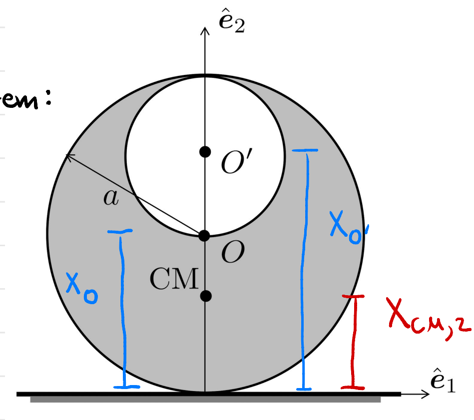
where C is either the CM or $v_C = 0$. As there is no obvious point with $v_C = 0$, we set $C = CM$. Now we can calculate everything we need for the kinetic energy.

- The total mass is the mass of the solid cylinder M_s minus the mass of the hole M_h.

$$M = M_s - M_h = \rho h\left[\pi a^2 - \pi\left(\tfrac{1}{2}a\right)^2\right] = \frac{3}{4}\pi a^2 \rho h$$

- Now to calculate the moment of inertia I_{CM}, we need to know the location of the CM.

To simplify this we can set $\theta = 0$
Now we can use the formula for the centre of mass for a particle system:



$$\underline{r}_{CM} = \frac{1}{M} \sum_{i=1}^{n} m_i \underline{r}_i$$

As we chose $\theta = 0$, $x_{CM,1} = 0$ and

$$x_{CM,2} = \frac{x_O M_s - x_{O'} M_h}{M_s - M_h}$$

where $x_O = a$, $x_{O'} = \frac{3}{2} a$

Note that we subtract the hole as its mass is essentially negative

$$\Rightarrow \quad x_{CM,2} = \frac{5}{6} a$$

Now the moment of inertia of the solid cylinder w.r.t. the CM:

$$I_{OCM} = \frac{1}{2} \rho h \pi a^4 + \underbrace{\rho h \pi a^2 \left(\frac{1}{6} a\right)^2}_{\text{Steiner's theorem}} = \frac{19}{36} \rho h \pi a^4$$

And for the hole:

$$I_{O'CM} = \frac{1}{2} \rho h \pi \left(\frac{a}{2}\right)^4 + \rho h \pi \left(\frac{a}{2}\right)^2 \left(\frac{2}{3} a\right)^2 = \frac{41}{288} \rho h \pi a^4$$

So the total moment of inertia can be found as

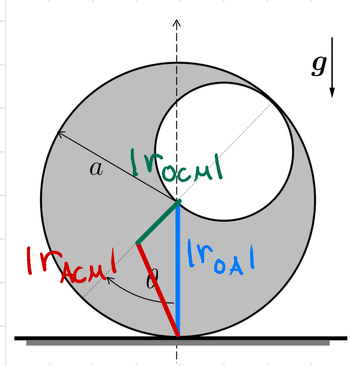
$$I_{CM} = I_{O,CM} - I_{O'CM} = \frac{37}{92} \rho h \pi a^4$$

- The last thing we need are $\underline{\omega}(t)$ and $v_{CM}(t)$. $\underline{\omega}(t)$ is simply

$$\underline{\omega}(t) = -\dot{\theta}(t)\, \underline{e}_3$$

Since the cylinder rolls without slipping, the ICR is point A and we can find

$$|\underline{v}_{CM}(t)| = \dot{\theta}(t)\, |r_{ACM}|$$



We can find $|\underline{r}_{ACM}|$ using the law of cosines:

$$|\underline{r}_{ACM}| = |r_{OCM}|^2 + |r_{OA}|^2 - 2|r_{OCM}||r_{OA}|\cos(\theta)$$

$$= \frac{1}{36}a^2 + a^2 - \frac{1}{3}a^2\cos(\theta)$$

$$\Rightarrow |v_{CM}(t)| = \dot{\theta}(t)\cdot a\sqrt{\frac{37}{36} - \frac{1}{3}\cos(\theta(t))}$$

- Using all of the above we can now calculate the kinetic energy:

$$\underline{\underline{T(t) = \pi\rho h a^4 \dot{\theta}(t)^2\left(\frac{37}{64} - \frac{1}{8}\cos(\theta(t))\right)}}$$

- Now we calculate the potential energy with A as reference point.

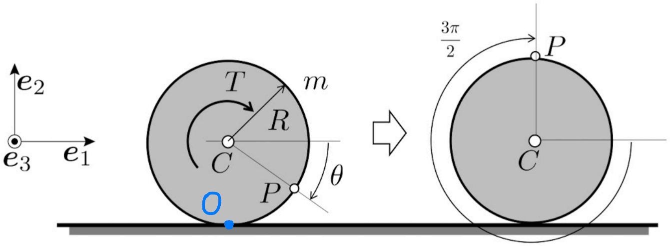
$$V = M\cdot g\cdot x_{CM,z} = \frac{3}{4}\rho h\pi a^2 g\,(a - |r_{OCM}|\cos(\theta(t)))$$

$$\underline{\underline{V = \frac{3}{4}\pi a^3\rho h g\left(1 - \frac{1}{6}\cos(\theta(t))\right)}}$$

Problem 2: Accelerated disk

A disk of mass m and radius R rolls without slipping on a horizontal floor. It is subject to a constant torque T, starting from rest at time $t = 0$. Assume that $\theta(t = 0) = 0$, where θ indicates the rotation angle of the disk, as indicated below.

Given: m, R, T



What is the acceleration $\boldsymbol{a}_P$ of point P on the wheel, when $\theta = \frac{3}{2}\pi$?

To find the angular acceleration $\ddot{\theta}$ of the disk we can use AMB, which we will evaluate with respect to the contact point O, for which we know $\underline{v}_O = \underline{0}$.

$$\dot{\underline{H}}_O + \cancel{\underline{v}_O}^{\,0} \times \underline{P} = \underline{M}_O^{ext}$$

$$\dot{\underline{H}}_O = -I_O \ddot{\theta}\, \underline{e}_3 , \quad I_O = \underbrace{\frac{1}{2} mR^2}_{I_C} + \underbrace{mR^2}_{\text{Steiner}} = \frac{3}{2} mR^2$$

$$\underline{M}_O^{ext} = -T \underline{e}_3$$

$$\Rightarrow \quad \frac{3}{2} mR^2 \ddot{\theta} = T$$

We can now use this to find $\ddot{\theta}$ and also integrate and use the initial conditions to find $\dot{\theta}$ and θ:

$$\Rightarrow \ddot{\theta} = \frac{2T}{3mR^2} \quad \Rightarrow \dot{\theta} = \frac{2T}{3mR^2} \cdot t \quad \Rightarrow \theta = \frac{T}{3mR^2} t^2$$

We are asked to find $\ddot{\theta}$ when $\theta = \frac{3}{2}\pi$ at time t_1, which is

$$\frac{3}{2}\pi = \frac{T}{3mR^2} t_1^2 \quad \Rightarrow \quad t_1 = \sqrt{\frac{9\pi m R^2}{2T}}$$

Now we need to find $\underline{a}_P$. The best option is to find a point with a known acceleration and then use the acceleration transfer formula. One such point is C with $\underline{a}_C = R\ddot{\theta}\underline{e}_1$, with which we can find:

$$\underline{a}_P = \underline{a}_C + \dot{\underline{\omega}} \times \underline{r}_{CP} + \underline{\omega} \times (\underline{\omega} \times \underline{r}_{CP}), \text{ where}$$

$$\underline{\omega} = -\dot{\theta}\,\underline{e}_3 \,,\ \dot{\underline{\omega}} = -\ddot{\theta}\,\underline{e}_3 \quad \text{and} \quad \underline{r}_{CP} = R\,\underline{e}_2$$

(Remember $\theta = \frac{3}{2}\pi$)

Now we can plug everything in:

$$\underline{a}_P = R\ddot{\theta}\,\underline{e}_1 + (-\ddot{\theta}\,\underline{e}_3) \times R\underline{e}_2 + (-\dot{\theta}\,\underline{e}_3) \times [(-\dot{\theta}\,\underline{e}_3) \times R\underline{e}_2]$$

$$= R\ddot{\theta}\,\underline{e}_1 + \ddot{\theta}R\,\underline{e}_1 - \dot{\theta}^2 R\,\underline{e}_2$$

$$= 2R\frac{2T}{3mR^2}\,\underline{e}_1 - R\left(\frac{2T}{3mR^2}t_1\right)^2\underline{e}_2$$

$$= 2R\frac{2T}{3mR^2}\,\underline{e}_1 - R\,\frac{4T^2}{9m^2R^4}\cdot\frac{9\pi mR^2}{2T}$$

$$\underline{a}_P = \frac{4T}{3mR}\,\underline{e}_1 - \frac{2\pi T}{mR}\,\underline{e}_1 = \underline{\underline{\frac{2T}{mR}\left(\frac{2}{3}\underline{e}_1 - \pi\,\underline{e}_2\right)}}$$