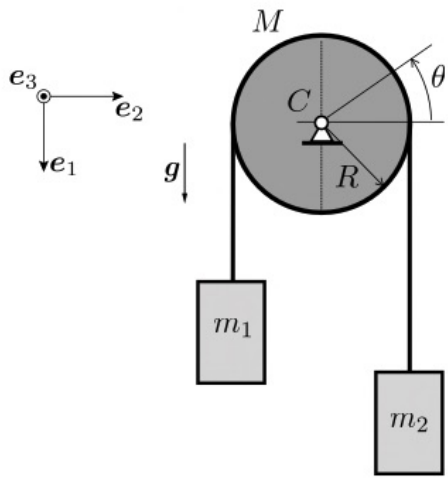
Problem 1: Atwood's machine

The system shown below consists of two particles of masses m_1 and m_2, attached to the ends of a string of length l, wrapped around a pulley[1]. The pulley is a uniform disk of radius R and mass M, which is hinged at point C. The string is taut, inextensible and massless. Moreover, assume that it does not slip between the string and the pulley. Gravity acts downwards, as shown.

Given: M, m_1, m_2, R, l, g



1. What is the acceleration $\ddot{x}_1$ of mass m_1?
2. What is the force in the string on the left side of the pulley?
3. What is the total energy of the system?

First let's take a look at the kinematics:

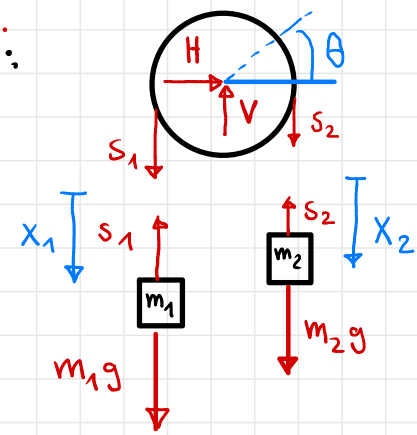
- The length of the string is constant:

$$x_1 + x_2 + \pi R = l = \text{const.}$$

- The string does not slip on the pulley:

$$\dot{x}_1 = R\dot{\theta} \quad \text{and} \quad \dot{x}_2 = -R\dot{\theta}$$

- Let us also draw an FBD ⟹



Now we can do LMB for the two blocks:

m_1: $\quad m_1\ddot{x}_1 = -S_1 + m_1 g$ positive by definition of x_1-direction ! $\quad$ (i)

m_2: $\quad m_2\ddot{x}_2 = -S_2 + m_2 g$ $\quad$ (ii)

For the pulley we can use AMB w.r.t. point C:

$$M_c = I_c \cdot \ddot{\theta}$$

where $\quad M_c = R(S_1 - S_2)$ (both forces cause a torque)

and $\quad I_c = \frac{1}{2} M R^2$ (moment of inertia of the disk)

$$\Rightarrow R(S_1 - S_2) = \frac{1}{2} M R^2 \ddot{\theta}$$

$$\Rightarrow \ddot{\theta} = \frac{2}{MR}(S_1 - S_2) \qquad \text{(iii)}$$

Using our kinematic relations we can now relate (i), (ii), and (iii) to $\ddot{x}_1$.

$$\ddot{x}_1 = -\ddot{x}_2 \quad , \quad \ddot{x}_1 = R\ddot{\theta}$$

$\Rightarrow$ (i) $\quad m_1\ddot{x}_1 = -S_1 + m_1 g \quad \Rightarrow S_1 = m_1(g - \ddot{x}_1)$

(ii) $\quad -m_2\ddot{x}_1 = -S_2 + m_2 g \quad \Rightarrow S_2 = m_2(g + \ddot{x}_1)$

(iii) $\quad \ddot{x}_1 = \frac{2}{M}(S_1 - S_2)$

$$\Rightarrow \ddot{x}_1 = \frac{2}{M}\left(m_1(g - \ddot{x}_1) - m_2(g + \ddot{x}_1)\right)$$

$$\ddot{x}_1 = \frac{2}{M}\left(g(m_1 - m_2) + \ddot{x}_1(-m_1 - m_2)\right)$$

$$\frac{1}{2}M\ddot{x}_1 + \ddot{x}_1(m_1 + m_2) = g(m_1 - m_2)$$

$$\Rightarrow \ddot{x}_1 = \frac{g(m_1 - m_2)}{m_1 + m_2 + \frac{1}{2}M}$$

which answers question 1), now to find S_1 we can simply use equation i) and plug in $\ddot{x}_1$

$$\Rightarrow S_1 = m_1 g - m_1 \ddot{x}_1$$

$$S_1 = m_1 g \frac{2m_1 + \frac{1}{2}M}{m_1 + m_2 + \frac{1}{2}M}$$

Now to calculate the total energy we can just sum over the kinetic and potential energies of all three bodies

this is negative as x_1 is defined downwards

$$E_1 = \frac{1}{2} m_1 \dot{x}_1^2 - m_1 g x_1$$

$$E_2 = \frac{1}{2} m_2 \dot{x}_2^2 - m_2 g x_2$$

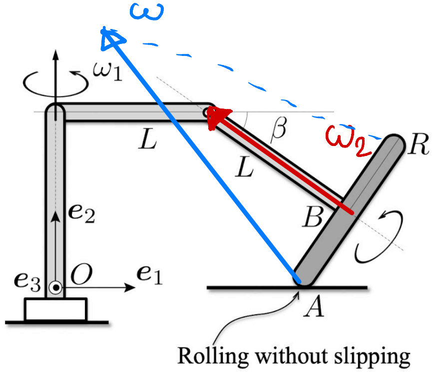
$$E_{disk} = \frac{1}{2} I_c \omega^2 = \frac{1}{2} \cdot \frac{1}{2} M R^2 \dot{\theta}^2$$ (this is the rotational kinetic energy!)

$$\Rightarrow E_{tot} = \frac{1}{2} R^2 \dot{\theta}^2 \left(m_1 + m_2 + \frac{1}{2}M\right) - m_1 g x_1 - m_2 g x_2$$

Problem 2: Disk on a shaft

A disk of radius R rolls without slipping over a horizontal surface while simultaneously rotating about an angled shaft. This angled shaft in turn rotates about a fixed, vertical axis. The angled arm of the shaft forms an angle β with the horizontal direction, while L denotes the length of the horizontal and tilted arms.

<u>Given:</u> R, L, β, ω_1



If the shaft is rotating with an angular velocity of magnitude ω_1, what is the angular velocity vector $\boldsymbol{\omega}$ of the disk in the shown configuration (i.e., when the frame lays in the plane spanned by $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$)?

The angular velocity of the disk is given by the sum of two contributions:

$$\underline{\omega} = \underline{\omega}_1 + \underline{\omega}_2$$

As $\underline{\omega}_1$ is know, we need to find $\underline{\omega}_2$

First we can use the velocity transfer formula:

$$\underline{V}_B = \underline{V}_0 + \underline{\omega}_1 \times \underline{r}_{OB} = -\omega_1 (L + L\cos(\beta))\, \underline{e}_3$$

Note that we don't actually need to carry out this cross product but can find $\underline{V}_B$ by using geometry.

We can now find $\underline{V}_A$ in a similar way:

$$\underline{V}_A = \underline{V}_B + \underbrace{\underline{\omega}}_{} \overset{= \underline{\omega}_1 + \underline{\omega}_2}{} \times \underline{r}_{BA}$$

where $\underline{\omega}_2 = -\omega_2 \cos(\beta)\, \underline{e}_1 + \omega_2 \sin(\beta)\, \underline{e}_2$

$$\underline{\omega}_1 = \omega_1 \underline{e}_2$$

$$\underline{r}_{BA} = -R\sin(\beta)\,\underline{e}_1 - R\cos(\beta)\,\underline{e}_2$$

$$\Rightarrow \underline{V}_A = \left[\omega_1\left(R\sin(\beta) - L - L\cos(\beta)\right) + \omega_2 R\right]\underline{e}_3$$

As we have rolling without slipping we know that $\underline{v}_A \overset{!}{=} \underline{0}$

$$\Rightarrow \quad \omega_2 = \frac{L + L\cos(\beta) - R\sin(\beta)}{R}\,\omega_1$$

we can now substitute this into $\underline{\omega}_2$ and then find

$$\underline{\omega} = \underline{\omega}_1 + \underline{\omega}_2$$

$$= -\omega_1 \frac{L + L\cos(\beta) - R\sin(\beta)}{R}\cos(\beta)\,\underline{e}_1$$

$$+ \omega_1\left(1 + \frac{L + L\cos(\beta) - R\sin(\beta)}{R}\sin(\beta)\right)\underline{e}_2$$

A more intuitive approach is to consider the velocities ω_1 and ω_2 induce in point A. They have to cancel each other out, so $V_A = 0$

$$V_{A,\omega_1} = \omega_1\left(L + L\cos(\beta) - R\sin(\beta)\right)$$

$$V_{A,\omega_2} = R\omega_2$$

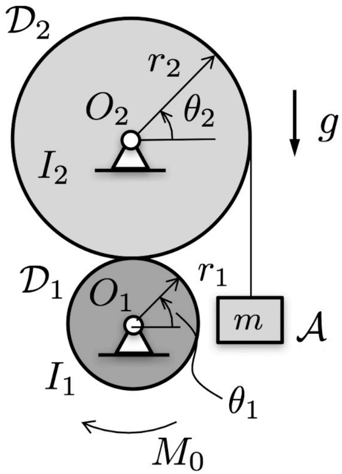
$$V_{A,\omega_1} \overset{!}{=} V_{A,\omega_2} \Rightarrow \omega_2 = \frac{L + L\cos(\beta) - R\sin(\beta)}{R}$$

And now we can use geometry to find the direction of $\underline{\omega}_2$, and the calculate $\underline{\omega} = \underline{\omega}_1 + \underline{\omega}_2$.

Problem 3: Disk pair

Two disks $\mathcal{D}_1$ and $\mathcal{D}_2$, of radii r_1 and r_2 and centroidal moments of inertia I_1 and I_2, are hinged at points O_1 and O_2, respectively. Rolling without slipping is enforced at their point of contact. A block $\mathcal{A}$ of mass m is attached to an inextensible, massless string that wraps around $\mathcal{D}_2$, and can move only in the vertical direction. A constant torque M_0 is applied to $\mathcal{D}_1$, as shown. Gravity acts downwards, as indicated, and the inextensible string is taut at all times.

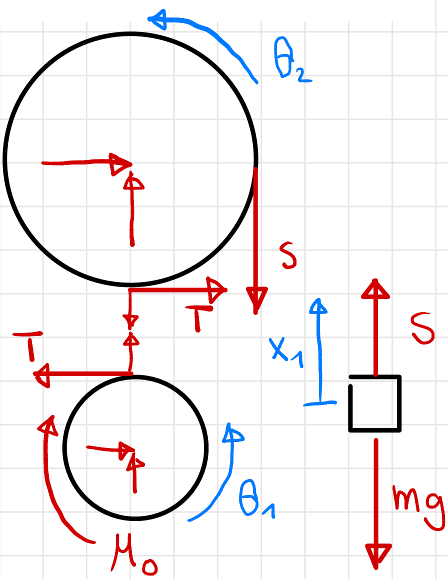
<u>Given:</u> r_1, r_2, I_1, I_2, m, M_0, g



What is the acceleration of block $\mathcal{A}$?

First we can draw a FBD, the we can relate all of the kinematic variables:

$$x = \theta_2 r_2 \quad , \quad \theta_1 r_1 = -\theta_2 r_2$$



Then we can use AMB and LMB to find one equation for each body:

$$D_1: \quad I_1 \ddot{\theta}_1 = -M_0 + T r_1$$

$$D_2: \quad I_2 \ddot{\theta}_2 = T r_2 - S r_2$$

$$m: \quad m\ddot{x} = S - mg$$

Now we can use our kinematic relations and find $\ddot{x}$ as the acceleration for the block

$$\ddot{x} = \frac{\frac{M_0}{r_1} - mg}{m + \frac{I_1}{r_1^2} + \frac{I_2}{r_2^2}}$$