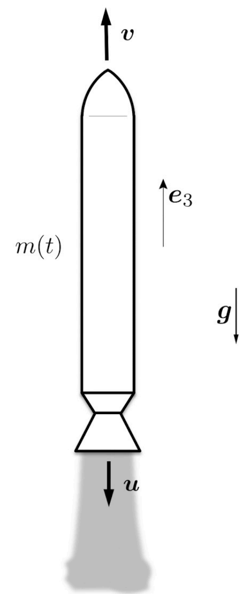
Problem 1: Rocket launch

A rocket of initial mass m_i is launched from the earth at rest and accelerates until all of its fuel has been exhausted. The residual (final) mass of the rocket's structure is m_f. Assume that the fuel is burned at a constant positive rate α and the exhaust gas is released vertically downward with a *relative* speed u = const. Neglect air resistance and assume the gravitational acceleration g (acting downwards) is constant.

Given: m_i, m_f, α, u, g



1. What is the velocity of the rocket $v(t)$ at a given time t after launch?
2. What altitude $x_3(t_f)$ has the rocket reached, when all its fuel has been burned at time t_f?

1) Let's use the rocket equation in $\underline{e}_3$:

$$F_{ext} = m\frac{dv}{dt} + \frac{dm}{dt}\cdot(v - v_m) \qquad (1)$$

where $F_{ext} = -mg$ and $(v - v_m) = u$ (relative)

- We can find $m(t) = m_i - \alpha t$
- And the time at which all fuel is burned:

$$t_f = \frac{m_i - m_f}{\alpha}$$

- We can now rearrange (5) by dividing by m and multiplying with dt

$$\Rightarrow \quad dv + u\frac{dm}{m} = -g\,dt$$

now we can integrate:

$$\int_0^v dv + u\int_{m_i}^{m(t)} \frac{dm}{m} = -g\int_0^t dt$$

$$v + u\Big[\ln(m)\Big]_{m_i}^{m(t)} = -gt$$

$$\boxed{\int \frac{1}{x}\,dx = \ln(x)}$$

$$\underline{\underline{v(t) = -gt - u\ln\left(\frac{m_i - \alpha t}{m_i}\right)}}$$

$$\boxed{\ln(a) - \ln(b) = \ln\left(\frac{a}{b}\right)}$$

2) We can integrate $v(t)$ from $t=0$ to $t=t_f$

$$X_3(t_f) = -\frac{1}{2}g t_f^2 - u\int_0^{t_f} \ln\left(\frac{m_i-\alpha t}{m_i}\right) dt$$

- Now we substitute $\mu = \dfrac{m_i - \alpha t}{m_i}$, $\Rightarrow dt = -\dfrac{m_i}{\alpha}\,d\mu$
- Now we can integrate:

remember to also change the boundries

$$u\int_0^{t_f} \ln\left(\frac{m_i-\alpha t}{m}\right) dt = u\int_1^{\frac{m_f}{m_i}} \ln(\mu)\,-\frac{m_i}{\alpha}\,d\mu$$

$$\Rightarrow \quad u \cdot \frac{-m_i}{\alpha}\Big[\mu \ln(\mu) - \mu\Big]_1^{\frac{m_f}{m_i}}$$

$$= u \cdot \frac{-m_i}{\alpha}\left[\frac{m_f}{m_i}\ln\left(\frac{m_f}{m_i}\right) - \frac{m_f}{m_i} + 1\right]$$

$$= u \cdot \frac{1}{\alpha}\left[\frac{m_f - m_i}{m_i} - \frac{m_f}{m_i}\ln\left(\frac{m_f}{m_i}\right)\right]$$

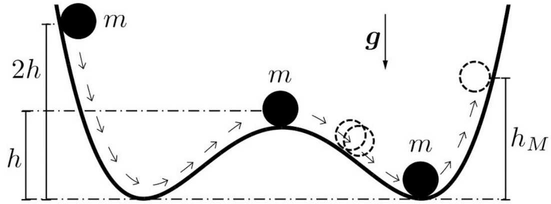
$$\Rightarrow x_3(t_f) = -\frac{1}{2} g\left(\frac{m_i - m_f}{\alpha}\right)^2$$

$$\boxed{- \frac{u}{\alpha}\left[\frac{m_f - m_i}{m_i} - \frac{m_f}{m_i}\ln\left(\frac{m_f}{m_i}\right)\right]}$$

Problem 2: Chain reaction

Three particles of equal masses m and negligible dimensions are at rest on a frictionless curvilinear guide lying in the vertical plane, as shown below (the black particles). The first particle is released from rest from an initial height $2h$ and eventually impacts the second particle with a *perfectly plastic* collision. After the impact, these two particles slide together, until they collide with the third particle. The coefficient of restitution of this last impact is a known value e. What is the maximum height h_M reached by the last particle? Assume that the particles never detach from the guide and that gravity acts downwards, as shown.

Given: h, e, m, g



We can use WEB and the collision equations to solve this problem.

- First we use WEB to find the velocity of the first particle at the first impact.

$$mgh = \frac{1}{2} m v_{1-}^2 \Rightarrow v_{1-} = \sqrt{2gh}$$

- Since the first impact is plastic $(e=0)$ the collision formula can be simplified to: ($v_{1+} = v_{2+}$, they stick !)

$$m v_{1-} + m v_{2-} = 2m v_{1+} \Rightarrow v_{1+} = v_{2+} = \frac{v_{1-}}{2}$$

- Now we can use WEB again to find v_1 before the second collision:

$$\frac{1}{2} 2m v_{1+}^2 + 2mgh = \frac{1}{2} 2m v_{1-}^2$$

$$\Rightarrow v_{1-} = \sqrt{v_{1+}^2 + 2gh}$$

- The collision formula gives the velocity of the 3rd particle after the 2nd collision:

$$V_{3+} = \frac{(1+e)\,2m}{m+2m}\,V_{1-}$$

- Now we can use WEB once again to find h_M

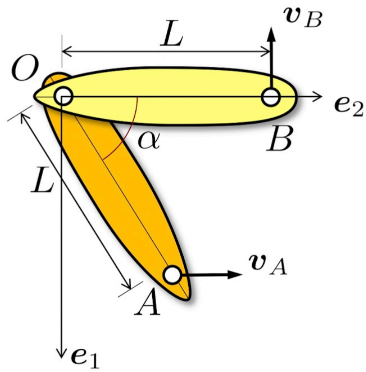
$$mgh_M = \frac{1}{2} m V_{3+}^2$$

$$\Rightarrow h_M = \frac{V_{3+}^2}{2g} = \frac{5(1+e)^2}{9} h$$

Problem 3: Kinematics of coupled rigid bodies

The system sketched below consists of two rigid bodies OA and OB, connected at point O by a joint that allows relative rotation. In the configuration shown, the two bodies enclose an angle α. The velocities at points A and B are given by $\boldsymbol{v}_A = v_A \boldsymbol{e}_2$ and $\boldsymbol{v}_B = -v_B \boldsymbol{e}_1$, respectively, and $|\boldsymbol{r}_{AO}| = |\boldsymbol{r}_{BO}| = L$. What is the magnitude ω_{OA} of the angular velocity $\boldsymbol{\omega}_{OA}$ of body OA?

Given: v_A, v_B, L, α



We can use the velocity transfer formula to find $\underline{v}_O$, from both bodies:

$$\underline{v}_O = \underline{v}_A + \underline{\omega}_{OA} \times \underline{r}_{AO} = \underline{v}_B + \underline{\omega}_{OB} \times \underline{r}_{BO}$$

which we can write as:

$$v_A \underline{e}_2 + \omega_{OA} \underline{e}_3 \times (-L\sin(\alpha)\,\underline{e}_1 - L\cos(\alpha)\,\underline{e}_2) =$$
$$-v_B \underline{e}_1 + \omega_{OB} \underline{e}_3 \times (-L\,\underline{e}_2)$$

the we can carry out the vector products:

$$v_A \underline{e}_2 - \omega_{OA} L \sin(\alpha)\,\underline{e}_2 + \omega_{OA} L\cos(\alpha)\,\underline{e}_1 =$$
$$-v_B \underline{e}_1 + \omega_{OB} L\,\underline{e}_1$$

which, whe we write it componentwise gives us two equations:

x_1: $-v_B + \omega_{OB} L = \omega_{OA} L \cos(\alpha)$

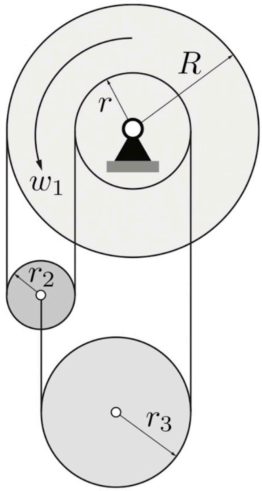
x_2: $v_A - \omega_{OA} L \sin(\alpha) = 0$

$$\Rightarrow \underline{\underline{\omega_{OA} = \frac{v_A}{L\sin(\alpha)}}}$$

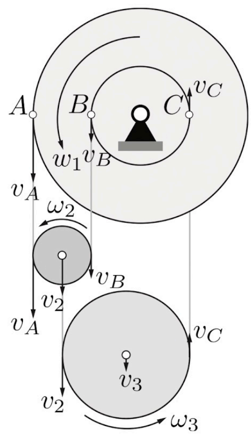
Problem 4: Pulley mechanism

As part of a pulley mechanism, three disks are connected to each other by non-slipping ropes, as shown below. What are the velocities of the centers of the two smaller rolls, if the large disk rotates with an angular velocity ω_1, as shown? What are their angular velocities?

Given: r, R, ω_1



We introduce points A, B, C to more easily convert between the velocities:



- $V_A = R\omega_1$, $V_B = r\omega_1$, $V_C = -r\omega_1$ (1)
- As there is no slip, we know for the smallest roll

$$V_A = V_2 + r_2\omega_2$$

This is essentially the transfer formula in 2D

$$V_B = V_2 - r_2\omega_2 \quad (2)$$

- We can now use (1), (2) and $r_2 = \frac{R-r}{2}$ to find ω_2:

$$\omega_2 = \frac{V_A - V_2}{r_2} = \frac{R\omega_1 - \frac{1}{2}(R+r)\omega_1}{\frac{1}{2}(R+r)} = \omega_1$$

For the bottom roll we can do the same:

$$V_3 = V_2 - r_3\omega_3 \ , \quad V_c = V_3 - r_3\omega_3$$

We also know $r_3 = \dfrac{R+3r}{4}$, so we can find ω_3

$$\omega_3 = \frac{V_2 - V_3}{r_3} = \frac{\frac{1}{2}(R+r)\omega_1 - \frac{1}{4}(R-r)\omega_1}{\frac{1}{4}(R+3r)} = \omega_1$$