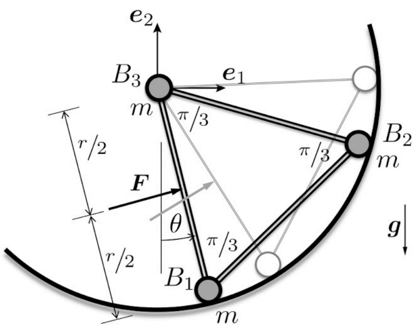
Problem 1: Particles on a track

Three particles B_1, B_2 and B_3 of equal masses m are connected to each other by massless rods of length r to form a rigid triangular unit. The system is supported in the vertical plane by a smooth, frictionless circular track of radius r. A force $\boldsymbol{F}$ of constant magnitude is applied perpendicular to one rod at its midpoint, as shown. This force remains perpendicular to the rod throughout the motion. Denote with θ the rotation of the system against the vertical axis, as indicated. Gravity acts downwards. You may assume that the particles remain on the track at all times.

Given: m, r, g



1. If the system starts from rest at $\theta = 0$, determine the minimum force magnitude F_{min} that is sufficient to bring the system to $\theta = \frac{\pi}{3}$.
2. Determine the speed v of particle B_1 when $\theta = \frac{\pi}{3}$ if $F = \frac{18}{\pi} mg$.

1) We want the system to be at rest again at $\theta = \pi/3$.

- Let's use WEB: $T(t_2) - T(t_1) = W_{12}$
- $T(t_2) = T(t_1) = 0$, as start and end at rest
- We can decompose the work W_{12} into a conservative and non-conservative part. In the conservative part we have gravity. The only non-conservative force is $\underline{F}$. It is non-conservative as the work it does is path-dependant.

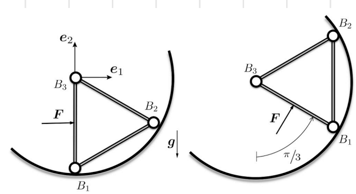


Figure 1: Initial and final state of the system.

- $W_{12}^{cons} = V_1 - V_2$

$$V_1 = \underbrace{0}_{B_1} + \underbrace{\tfrac{1}{2} rmg}_{B_2} + \underbrace{rmg}_{B_3}$$

$$V_2 = \underbrace{\frac{1}{2} rmg}_{B_1} + \underbrace{\frac{3}{2} rmg}_{B_2} + \underbrace{rmg}_{B_3}$$

$$\Rightarrow W_{12}^{cons} = -\frac{3}{2} mgr$$

- $$W_{12}^{non\text{-}cons.} = \int_{\underline{r}_1}^{\underline{r}_2} \underline{F} \cdot d\underline{r}$$

As $\underline{F}$ always points in direction of the trajectory, we can see that $\underline{F}$ can be seen as a constant in this integral $\Rightarrow$ pull it out of the integral

$$= F \cdot \underbrace{\int_{\underline{r}_1}^{\underline{r}_2} d\underline{r}}$$

$\Rightarrow$ This is just the pathlength

$$\Rightarrow W_{12}^{non\text{-}cons} = F \cdot \frac{r}{2} \cdot \frac{\pi}{3}$$

- Now we can plug this into WEB:

$$W_{12} = W_{12}^{cons} + W_{12}^{non\text{-}cons} = 0$$

$$\Rightarrow F \frac{\cancel{r}\pi}{6} - \frac{3}{2} mg\cancel{r} = 0$$

$$\underline{\underline{F_{min} = \frac{9}{\pi} mg}}$$

Note: This force is not enough to keep the system at $\theta = \pi/3$!
(This can be verified with AMB)

2) We can once again use WEB, just that now $T(t_2) \neq 0$

- $$W_{12}^{non\text{-}cons} + W_{12}^{cons} = T_2$$

- $$T_2 = 2 \cdot \frac{1}{2} mv^2$$

Two particles move

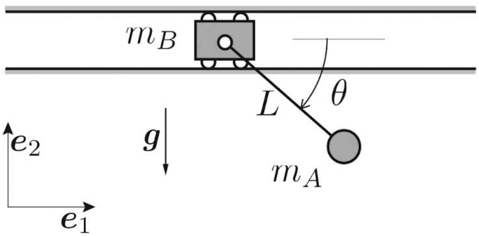
$\Rightarrow$ plug in the given value for F $\Rightarrow$ $\underline{\underline{v = \sqrt{\frac{3gr}{2}}}}$

Problem 2: Trolley on a guide

A particle of mass m_A is connected to a trolley B of mass m_B by a massless rod of length L. The trolley slides without friction along a horizontal guide, and the whole system lies in the vertical plane with gravity acting downwards, as shown. The system is released from rest at $\theta = 0$.

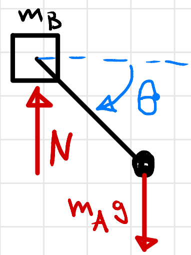
Determine the speeds v_A and v_B of A and B, respectively, when angle θ reaches $\pi/2$ (for the first time).

<u>Given:</u> L, m_A, $m_B = 3m_A$, g



As we are looking for v_A <u>and</u> v_B we need two equations.

Let's begin with LMB of the system:



$$M(a_{CM,1}\underline{e_1} + a_{CM,2}\underline{e_2}) = [N - (m_A + m_B)g]\,\underline{e_2}$$

As there are no external forces in $\underline{e_1}$-direction, this means linear momentum is conserved in $\underline{e_1}$.

$$\Rightarrow \quad m_A v_{A,1} + m_B v_{B,1} = 0$$

the system starts from rest

$$\Rightarrow \quad v_{B,1} = -\frac{m_A}{m_B} v_{A,1} \qquad (i)$$

Now we need a second equation, for which we use WEB

$$T(t_1) = 0, \quad T(t_2) = \frac{1}{2} m_A v_A^2 + \frac{1}{2} m_B v_B^2$$

$$V_1 - V_2 = m_A g L$$ (as m_A swings down)

$$\Rightarrow \quad \frac{1}{2} m_A v_A^2 + \frac{1}{2} m_B v_B^2 = m_A g L \qquad (ii)$$

Now we have two equations (i) and (ii) which we can solve for v_A, v_B:

$$v_A = \pm \sqrt{\frac{3gL}{2}}$$

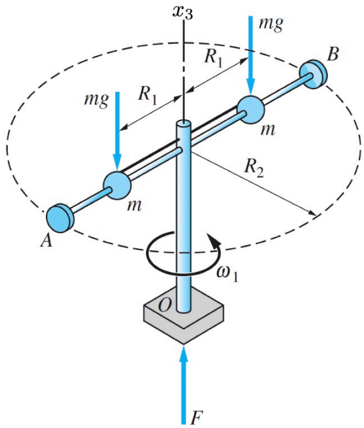
, we consider the first time the mass swings down
$\Rightarrow$ the sign is negative

$$\Rightarrow v_A = -\sqrt{\frac{3gL}{2}}, \quad v_B = \frac{1}{3}\sqrt{\frac{3gL}{2}}$$

Problem 3: Rotating particles

The assembly shown in the figure below consists of two particles, each of mass m, which slide on a frictionless, rigid bar AB of negligible mass. The support at O permits free rotation of the frame about the x_3-axis. The frame is initially rotating with the angular velocity ω_1 while strings hold the particles at the radial distance R_1. The strings are then cut simultaneously, permitting the particles to slide toward the stops at A and B, which are located at the radial distance R_2. Assume that the particles do not rebound after striking the stops but simply come to rest there. What is the angular velocity ω_2 of the system, once the particles have reached positions A and B? Gravity acts downwards.

Given: R_1, R_2, ω_1, m, g



Let's use AMB: $\underline{M}_B = \dot{\underline{H}}_B + \underline{v}_B \times \underline{p}$

$\underline{M}_B = \underline{0}$, $\underline{v}_B = \underline{0}$

$\Rightarrow$ Angular momentum is conserved, we can compare states:

$$\underline{H}(t_1) = 2 \cdot m R_1^2 \cdot \omega_1 \, \underline{e}_3$$

$$\underline{H}(t_2) = 2 \cdot m R_2^2 \cdot \omega_2 \, \underline{e}_3$$

$$H(t_1) \overset{!}{=} H(t_2)$$

$$\Rightarrow \omega_2 = \left(\frac{R_1}{R_2}\right)^2 \omega_1$$

Note: kinetic energy is not conserved, as we perform work to change the particles position: