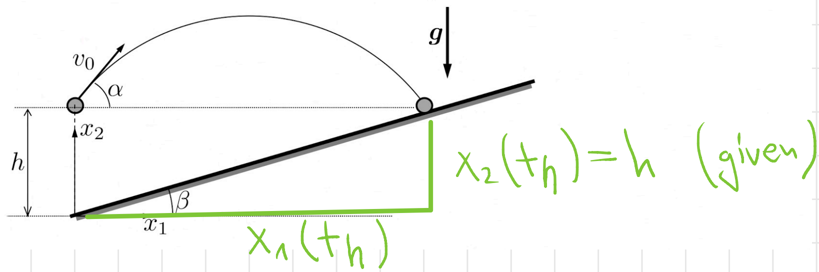
Problem 1: Particle trajectory

A particle is launched at time $t = 0$ from position $\boldsymbol{r}(0) = h\boldsymbol{e}_2$ with an initial velocity v_0 under an angle α with the horizontal axis, as shown. During its flight, the particle is subject to a constant acceleration $\boldsymbol{a} = -g\boldsymbol{e}_2$. What is the value of the inclination angle β, such that the particle eventually hits the inclined ground at the same height h, as shown in the figure?

Given: v_0, α, h, g



- We need to find out, at which time t_h the particle is at height $x_2(t_h) = h$.
- Then we can find $x_1(t_h)$, and use $x_1(t_h)$ and $x_2(t_h)$ to find the angle β.

- First, let us compute $x_1(t)$ and $x_2(t)$. We do this by twice integrating $\ddot{x}_1$, $\ddot{x}_2$ with respect to time.

- Let's first find $x_1(t)$:

$$\ddot{x}_1 = 0, \quad \dot{x}_1(0) = v_0 \cdot \cos(\alpha)$$

$$\Rightarrow \dot{x}_1(t) = \dot{x}_1(0) + \int_0^t \ddot{x}_1 dt = v_0 \cdot \cos(\alpha)$$

$$\dot{x}_1 = v_0 \cos(\alpha), \quad x_1(0) = 0$$

$$\Rightarrow x_1(t) = x_1(0) + \int_0^t \dot{x}_1 dt = v_0 \cdot \cos(\alpha) \cdot t$$

And now equally for $x_2(t)$

$$\ddot{x}_2 = -g, \quad \dot{x}_2(0) = v_0 \sin(\alpha)$$

$$\Rightarrow \dot{x}_2 = \dot{x}_2(0) + \int_0^t \ddot{x}_2 \, dt = v_0 \sin(\alpha) - gt$$

$$\dot{x}_2 = v_0 \sin(\alpha) - gt, \quad x_2(0) = h$$

$$\Rightarrow x_2 = x_2(0) + \int_0^t \dot{x}_2 \, dt = h + v_0 \sin(\alpha) t - \frac{1}{2} g t^2$$

- Now we know $x_2(t)$, we can find the time t_h, at which the particle hits the plane

$$x_2(t = t_h) = h$$

$$h + v_0 \sin(\alpha) t_h - \frac{1}{2} g t_h^2 = h \qquad | -h \quad | : t_h$$

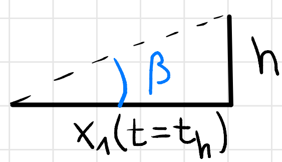
$$v_0 \sin(\alpha) - \frac{1}{2} g t_h = 0 \qquad | + \frac{1}{2} g t_h \quad | \cdot 2 \quad | : g$$

$$t_h = \frac{2 v_0 \sin(\alpha)}{g}$$

- Now we can find $x_1(t = t_h)$

$$x_1(t = t_h) = v_0 \cos(\alpha) \cdot \frac{2 v_0 \sin(\alpha)}{g} = \frac{2 v_0^2 \sin(\alpha) \cos(\alpha)}{g}$$

- Now we can find β using geometry:



$$\beta = \arctan \frac{h}{x_1(t = t_h)}$$

$$= \arctan \left(\frac{gh}{2 v_0^2 \sin(\alpha) \cos(\alpha)} \right)$$

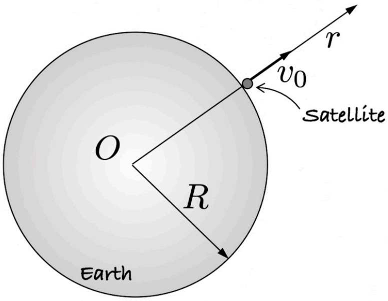
Problem 2: Orbital dynamics

A satellite is launched from the surface of the Earth with an initial radial velocity v_0, and it is subject to the gravitational radial acceleration $a_r = -g\frac{R^2}{r^2}$, where R is the radius of the Earth and r the distance from the center of the Earth. Neglect the Earth's rotation.

Given: R, g

(a) What is the velocity $v(r)$ of the satellite (for a given v_0)?

(b) What is the minimum initial velocity v_0, so that the satellite can leave the Earth's gravitational influence (i.e., travel to infinite distances)?



a) a_r depends on r, not on t, that's why we have to use a little trick:

$$\boxed{a_r = \frac{dv}{dt} = \frac{dv}{dt} \cdot \frac{dr}{dr} = \frac{dv}{dr} \cdot \frac{dr}{dt} = \frac{dv}{dr} \cdot v}$$

Now we can plug $a_r = -g\frac{R^2}{r^2}$ into this:

$$\Rightarrow -g\frac{R^2}{r^2} = \frac{dv}{dr} \cdot v \quad | \cdot dr$$

(This is essentially separation of variables)

$$-g\frac{R^2}{r^2} \cdot dr = v \cdot dv$$

Now we can integrate both sides, note the integration bounds:

$$\Rightarrow -gR^2 \int_R^r \frac{1}{r^2}\,dr = \int_{v_0}^{v} v\,dv$$

$$\boxed{\int \frac{1}{r^2}\,dr = -\frac{1}{r}}$$

$$\Rightarrow -gR^2\left(\frac{1}{R}-\frac{1}{r}\right) = \frac{1}{2}\left(v^2 - v_0^2\right)$$

note the sign!

$$gR^2\left(\frac{R-r}{rR}\right) = \frac{1}{2}\left(v^2 - v_0^2\right)$$

$$v_0^2 + 2gR\left(\frac{R-r}{r}\right) = v^2 \Rightarrow v(r) = \sqrt{v_0^2 - 2gR\,\frac{R-r}{r}}$$

b) $v(r)$ needs to be > 0 for $r \to \infty$, so we can ensure the satellite always moves away from earth

$$v(r \to \infty) = \lim_{r\to\infty} \sqrt{v_0^2 + 2gR^2\left(\cancel{\frac{1}{r}}^{\,0} - \frac{1}{R}\right)}$$

This is a different form of the result from a)

$$= \sqrt{v_0^2 - 2gR} \overset{!}{>} 0$$

$$\Rightarrow v_0 > \sqrt{2gR}$$

Hints for the Homework

Problem 3

a) Remember how $\underline{v}$ and $\underline{a}$ are defined in a polar frame

velocity and acceleration components in **polar coordinates** (r, φ):

$$\boldsymbol{v} = \dot{r}\boldsymbol{e}_r + r\dot{\varphi}\boldsymbol{e}_\varphi, \qquad \boldsymbol{a} = (\ddot{r} - r\dot{\varphi}^2)\boldsymbol{e}_r + (2\dot{r}\dot{\varphi} + r\ddot{\varphi})\boldsymbol{e}_\varphi$$

b) Integrate $\dot{\varphi}$ to find φ, then solve for $t(\varphi)$ and use $t(\varphi)$ in $r(t)$

c) same as a), use definition

Problem 4

- Express the distance travelled as a function of φ
- Use $v = \dot{x} = \text{const.}$
- $1 + \tan^2(x) = \dfrac{1}{\cos^2(x)}$