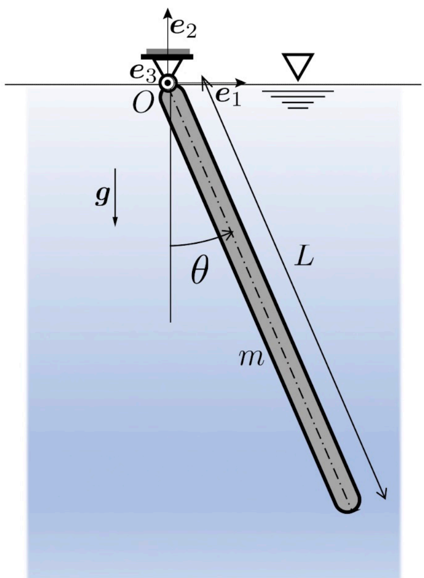
Problem 1: Vibration in a fluid

A uniform slender bar of mass density ρ, length L, width b, and out-of-plane thickness t is hinged at its tip, and swings in a fluid. The viscous fluid applies a pressure $\boldsymbol{p}(x)$ to the bar, which is proportional to the velocity $\boldsymbol{v}(\boldsymbol{x})$ of the bar with a proportionality factor c (thus acting like many distributed dampers onto the bar). Consider small vibrations about $\theta = 0$. Gravity acts downwards, as shown.

Given: ρ, L, b, t, c, $\boldsymbol{p}(x) = c\,\boldsymbol{v}(x)$, $|\theta(t)| \ll 1$, g



1. What is the equation of motion of the vibrating bar?

2. What is the period of vibration of the bar (assuming low damping)?

1) To find the equation of motion we can either use AMB or Lagrange. In this case AMB is a bit quicker:

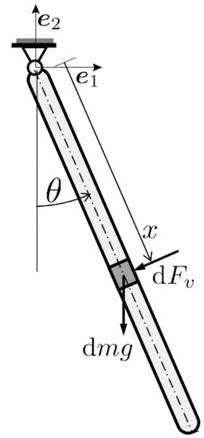
$$\text{AMB:} \quad \dot{\underline{H}}_0 = \underline{M}_0$$

$$\text{where} \quad \underline{H}_0 = I_0 \underline{\omega} = \frac{1}{3} m L^2 \dot{\theta}\, \underline{e}_3$$

$$\text{and thus} \quad \dot{\underline{H}}_0 = \frac{1}{3} m L^2 \ddot{\theta}\, \underline{e}_3$$

Now we need to find the external torques. As the force is spread out over the beam this is a bit tricky.

- That's why at first we only look at a small section of the beam with Length dx (and still thickness t).
  The force acting on this element is composed of the viscous damping and gravity:



$$d\underline{F} = d\underline{F}_v + d\underline{F}_g$$

- Let's first look at the viscous damping, which is

$$d\underline{F}_v = -\underline{p}(x) \cdot t \cdot dx \quad \text{(opposing the motion)}$$

$$\text{Force} = \text{pressure} \cdot \text{Area}$$

Where we use the given $\underline{p}(x) = c \cdot \underline{v}(x)$
For this we need $\underline{v}(x)$ which we can find as the derivative of $\underline{r}(t)$

$$\underline{r}(t) = x\left(\sin(\theta)\underline{e}_1 - \cos(\theta)\underline{e}_2\right)$$

$$\Rightarrow \underline{v}(t) = x\dot{\theta}\left(\cos(\theta)\underline{e}_1 + \sin(\theta)\underline{e}_2\right)$$

The other component of the force is

$$d\underline{F}_g = -dm \cdot g \cdot \underline{e}_2 = -\rho t b dx \cdot g \cdot \underline{e}_2$$

- Now that we have our force we can find the torque $d\underline{M}$ acting on our element

$$d\underline{M} = \underline{r}(x) \times \left(d\underline{F}_v + d\underline{F}_g\right)$$

- Now we already see that this will contain, so we can use the small angle approximation which gives

$$\underline{v}(x) = x\dot{\theta}\,\underline{e}_1 + x\theta\dot{\theta}\,\underline{e}_2$$

$$\underline{r}(x) = x\theta\,\underline{e}_1 - x\underline{e}_2$$

$$\Rightarrow d\underline{M}_0 = \left(x\theta\,\underline{e}_1 - x\underline{e}_2\right) \times \left(-cx\dot{\theta}\,\underline{e}_1 - \left(cx\theta\dot{\theta} + \rho b g\right)\underline{e}_2\right) t dx$$

which becomes

$$d\underline{M}_o = -\left(\underbrace{cx^2\theta^2\dot{\theta}}_{=0 \text{ due to the } \theta^2} + \rho b g x\theta + cx^2\dot{\theta}\right)\underline{e}_3$$

- Now to find the torque on the entire beam we can simply integrate:

$$\underline{M}_o = \int_0^L d\underline{M}_o = -\int_0^L (\rho b g \theta x + cx^2\dot{\theta})t\,dx\,\underline{e}_3$$

$$= -\left(mg\frac{L}{2}\theta + c\frac{L^3 t}{3}\dot{\theta}\right)\underline{e}_3$$ where we used $m = \rho t b L$

- Now we can plug this into AMB to find the equation of motion:

$$\Rightarrow \frac{1}{3}mL^2\ddot{\theta} + \frac{L^2 m}{3}\frac{c}{\rho b}\dot{\theta} + mg\frac{L}{2}\theta = 0$$

$$\Rightarrow \ddot{\theta} + \frac{c}{\rho b}\dot{\theta} + \frac{3g}{2L}\theta = 0$$

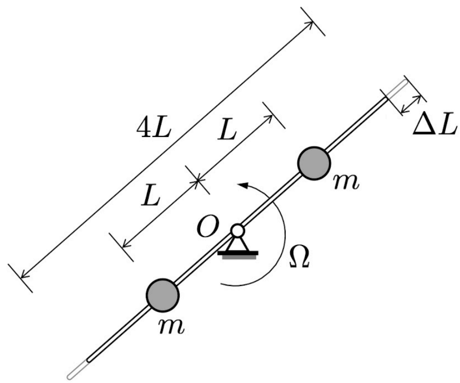
2) For small c we have an underdamped vibration:

$$T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{\omega_0^2 - \delta^2}} = \frac{2\pi}{\sqrt{\frac{3g}{2L} - \frac{c^2}{4\rho^2 b^2}}}$$

Problem 2: Rotating bar-particle system I

A rod of *negligible mass* and length $4L$ rotates about its midpoint O with a constant angular velocity Ω. The cross-sectional area of the bar is A, its Young's modulus is E. Two particles of equal masses m are attached to the bar at distances L from the the hinge O, as shown below. Neglect gravity.

Given: m, L, A, E, Ω



What is the elongation ΔL at the tip on each side of the bar?

We notice that only the first part of the beam $(0 < x < L)$ is stretched as there are no forces in the second part.

This means we can replace the first half of the beam with a spring of stiffness

$$k = \frac{F}{\Delta L} = \frac{EA}{L}$$

which we rearrange for ΔL

$$\Delta L = \frac{FL}{EA}$$

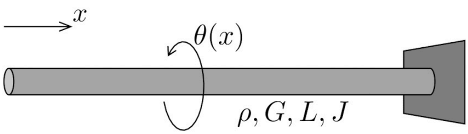
where F is the centripetal force caused by the rotating particle

$$F = m\Omega^2 L \Rightarrow \underline{\underline{\frac{m\Omega^2 L^2}{EA}}}$$

Problem 3: Vibration of a free-clamped torsional shaft

Consider a slender elastic torsional shaft of length L, shear modulus G, mass density ρ, and polar moment of inertia J. The shaft is clamped at $x = L$ and is free to rotate along its axis at $x = 0$, as shown below. Denote by $\theta(x,t)$ the twist angle of the cross-section at position x along the shaft at time t. Neglect gravity.

Given: ρ, G, L, J



What is the general solution $\theta(x,t)$ for free vibrations of the shaft? What is its fundamental frequency ω_0?

The equation of motion is, as given in the formula collection:

$$\ddot{\theta}(x,t) = c_T^2\, \theta_{xx}(x,t) \qquad \text{with} \qquad c_T = \sqrt{\frac{G}{\rho}}$$

To solve this we can use the separable Ansatz:

$$\theta(x,t) = \hat{\theta}(x) q(t)$$

with:

$$\hat{\theta}(x) = B_1 \cos\left(\frac{\omega}{c_T}x\right) + B_2 \left(\frac{\omega}{c_T}x\right)$$

$$q(t) = A_1 \cos(\omega t) + A_2 \cos(\omega t)$$

Now we can find B_1, B_2 using the given boundary conditions:

- The shaft must be torque free at $x=0$

$$\Rightarrow \theta_x(0,t) = 0 \quad \forall t$$

$$\Rightarrow \hat{\theta}_x(0) = B_2 \frac{\omega}{c_T} = 0 \Rightarrow B_2 = 0$$

• The clamped end must be at rest:

$$\Rightarrow \theta(L,t) = 0 \quad \forall\, t$$

$$\Rightarrow \hat{\theta}(L) = B_1 \cos\left(\frac{\omega}{c_T} L\right) = 0$$

As B_1 can't also be zero we follow that

$$\cos\left(\frac{\omega}{c_T} L\right) = 0 \Rightarrow \frac{\omega_n}{c_T} L = \frac{2n+1}{2}\pi \quad , n \in \mathbb{N}$$

So the eigenfrequencies are

$$\omega_n = \frac{(2n+1)\pi\, c_T}{2L}$$

and especially the fundamental frequency:

$$\omega_0 = \frac{\pi c_T}{2L}$$

and the general solution:

$$\theta(x,t) = \sum_{n=0}^{\infty} \cos\left(\frac{\omega_n}{c_T} x\right)\left[B_1 A_1 \cos(\omega_n t) + B_1 A_2 \sin(\omega_n t)\right]$$

which is the superposition of the solution of each eigenfrequency.