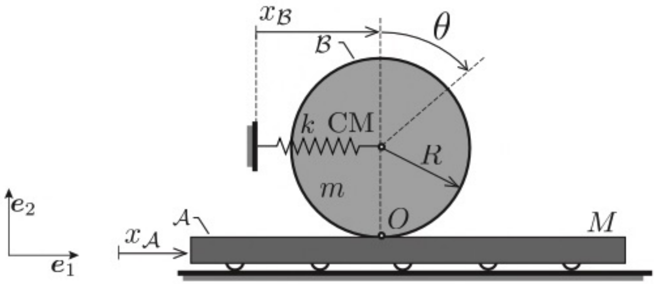
Problem 1: Disk on a carrier

The system shown below consists of a disk $\mathcal{B}$ of radius R and mass m, rolling without slipping on a block $\mathcal{A}$ of mass M, which is, in turn, supported to move horizontally. A spring of stiffness k and zero unstretched length connects a fixed wall with the center of mass CM of the disk. Denote by $x_{\mathcal{A}}$ and $x_{\mathcal{B}}$ the (absolute) translation of the block and the center of mass of the disk in $\mathbf{e}_1$-direction, respectively, and assume that $x_{\mathcal{A}} = x_{\mathcal{B}}$ if $\theta = 0$. Neglect gravity and assume that the disk does not interact with the fixed wall.

Given: m, k, M, R



What is the non-zero eigenfrequency of the system?

First we need to establish the kinematic relations for the system:
From the no-slip condition we get:

$$x_B = x_A + R\theta \quad \Rightarrow \quad \dot{x}_B = \dot{x}_A + R\dot{\theta} \quad \Rightarrow \quad \ddot{x}_B = \ddot{x}_A + R\ddot{\theta}$$

This means our system has only two degrees of freedom. We choose x_A and x_B as our generalized DOF.
Now we have to find our Lagrangian. The total kinetic energy is:

$$T = \frac{1}{2}M\dot{x}_A^2 + \frac{1}{2}m\dot{x}_B^2 + \frac{1}{2}\frac{1}{2}mR^2\dot{\theta}^2 = \frac{1}{2}M\dot{x}_A^2 + \frac{1}{2}m\dot{x}_B^2 + \frac{1}{4}m(\dot{x}_B - \dot{x}_A)^2$$

and the potential energy is:

$$V = \frac{1}{2}kx_B^2$$

There are no non-conservative forces.
Now we can use the Lagrange equations for x_A and x_B to find the EOM

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_A}\right) - \frac{\partial L}{\partial x_A} = 0 \quad \text{and} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_B}\right) - \frac{\partial L}{\partial x_B} = 0$$

$\Rightarrow$ i) $M\ddot{x}_A - \frac{1}{2}m(\ddot{x}_B - \ddot{x}_A) = 0$

ii) $m\ddot{x}_B + \frac{1}{2}m(\ddot{x}_B - \ddot{x}_A) + kx_B = 0$

Now to find the eigenfrequencies we first have to notice that we have one rigid body mode (zero eigenfrequency) where $x_B = \text{const.}$ and the wheel only rotates as the platform moves. This means to find the second eigenfrequency we can reduce both equations to one equation by rearranging (i):

$$\ddot{x}_A = \frac{m}{2M + m} \ddot{x}_B$$

which we plug into (ii) to find:

$$\frac{m^2 + 3Mm}{m + 2M} \ddot{x}_B + kx_B = 0$$

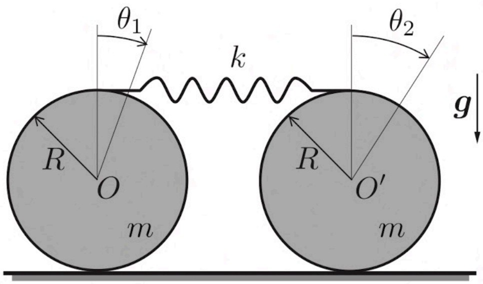
From this we can now just read the eigenfrequency as we usually do:

$$\underline{\underline{\omega = \sqrt{\frac{k(m+2M)}{m^2 + 3Mm}}}}$$

Problem 2: Connected disks

The system shown below consists of two identical disks of masses m and radii R, rolling without slipping on a horizontal surface. The two disks are connected by an elastic spring of stiffness k, which is unstretched when $\theta_1 = \theta_2$. Consider small rotations around $\theta_1 = \theta_2 = 0$. Gravity acts downwards, as indicated.

Given: m, k, R, g



What are the eigenfrequencies and eigenmodes of the system?

Let's use the system matrices to find the eigenmodes and eigenfrequencies. For that we need the kinetic and potential energy:

Steiner

$$T = \frac{1}{2}\frac{3}{2} mR^2 \left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right)$$

$$V = \frac{1}{2} k (2R^2)(\theta_2 - \theta_1)^2$$

Now we can calculate the mass matrix as:

$$\underline{\underline{M}} = \frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j} = \frac{3}{2} mR^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and the stiffness matrix is:

$$\underline{\underline{K}} = \frac{\partial^2 V}{\partial q_i \partial q_j} = 4kR^2 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

To make things easier we can divide both $\underline{\underline{M}}$ and $\underline{\underline{K}}$ by $\frac{3}{2}mR^2$. Now the characteristic equation follows as:

$$\det\left(\underline{\underline{K}} - \omega^2 \underline{\underline{M}}\right) = 0$$

$$\det\left[\frac{8k}{3m}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} - \omega^2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right] = 0$$

$$\det\begin{pmatrix}\frac{8k}{3m} - \omega^2 & -\frac{8k}{3m}\\ -\frac{8k}{3m} & \frac{8k}{3m} - \omega^2\end{pmatrix} = 0$$

$$\Rightarrow \left(\frac{8k}{3m} - \omega^2\right)^2 - \left(\frac{8k}{3m}\right)^2 = 0$$

$$\Rightarrow \left(\frac{8k}{3m}\right)^2 - \frac{16k}{3m}\omega^2 + \omega^4 - \left(\frac{8k}{3m}\right)^2 = 0$$

$$\Rightarrow \omega^2\left(\omega^2 - \frac{16k}{3m}\right) = 0$$

From this we the eigenfrequencies as:

$$\underline{\underline{\omega_1 = 0}} \quad \text{and} \quad \underline{\underline{\omega_2 = 4\sqrt{\frac{k}{3m}}}}$$

Now we can find both eigenmodes using $(\underline{\underline{K}} - \omega^2 \underline{\underline{M}})\underline{\hat{x}} = \underline{0}$

ω_1:

$$\Rightarrow \begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} \cdot \underline{\hat{x}}_1 = \underline{0} \Rightarrow \underline{\underline{\underline{\hat{x}}_1 = \begin{pmatrix}1\\ 1\end{pmatrix}}}$$

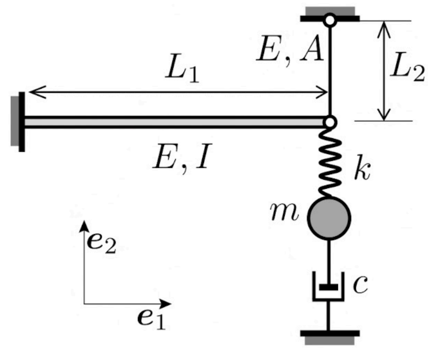
ω_2:

$$\Rightarrow \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \cdot \underline{\hat{x}}_2 = \underline{0} \Rightarrow \underline{\underline{\underline{\hat{x}}_2 = \begin{pmatrix}-1\\ 1\end{pmatrix}}}$$

Problem 3: Bar-particle system

The shown system consists of a particle of mass m, attached to a spring of stiffness k and a damper of viscous coefficient c. The spring is, in turn, connected to the tip of a cantilever beam of length L_1, Young modulus E and area moment of inertia I. Moreover, the tip of the beam is suspended by a bar of length L_2, cross-sectional area A and Young's modulus E. The mass of the beam and of the bar as well as gravity can be neglected.

<u>Given:</u> m, k, c, E, A, I, L_1, L_2, $EI/L_1^3 = EA/L_2 = k$



1. Assuming low damping, what is the eigenfrequency of the system?

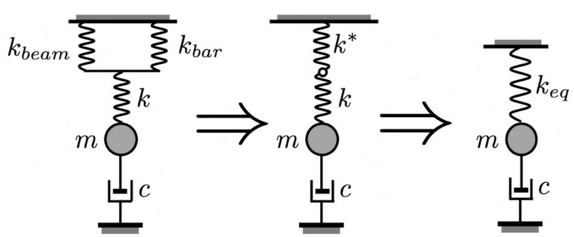
For the next task, denote the eigenfrequency of the system by ω_d.

2. If the system is released at time $t = 0$ from $x(0) = x_0$ and $\dot{x}(0) = 0$ (x being the vertical displacement of the particle), what is the subsequent motion of the particle?

1) In a first step we can simplify the spring system as pictured below

$$k^* = k_{beam} + k_{bar}$$

$$\frac{1}{k_{eq}} = \frac{1}{k} + \frac{1}{k^*}$$



Using the formula collection and the given relations we get

$$k_{beam} = \frac{3EI}{L_1^3} = 3k, \qquad k_{bar} = \frac{EA}{L} = k$$

Plugging this in we find k_{eq}:

$$k^* = 4k \Rightarrow k_{eq} = \frac{4}{5}k$$

Thus the equation of motion is:

$$m\ddot{x} + c\dot{x} + \frac{4}{5}kx = 0$$

From this we find

$$\delta = \frac{c}{2m} , \quad \omega_0^2 = \frac{4}{5}\frac{k}{m}$$

This means the damped eigenfrequency is:

$$\underline{\underline{\omega_d = \sqrt{\omega_0^2 - \delta^2} = \sqrt{\frac{4}{5}\frac{k}{m} - \left(\frac{c}{2m}\right)^2}}}$$

2) For the motion we use the general solution for an underdamped oscillator

$$x(t) = e^{-\delta t}\left[A\cos(\omega_d t) + B\sin(\omega_d t)\right]$$

We can find the constants using the initial conditions:

$$x(0) = A = x_0 \quad \text{and} \quad \dot{x}(0) = -\delta A + B\omega_d = 0$$

$$\Rightarrow A = x_0 \quad \text{and} \quad B = \frac{\delta A}{\omega_d} = \frac{\delta x_0}{\omega_d}$$

There fore the motion of the particle is:

$$\underline{\underline{x(t) = e^{-\delta t}\left[x_0 \cos(\omega_d t) + \frac{\delta x_0}{\omega_d}\sin(\omega_d t)\right]}}$$