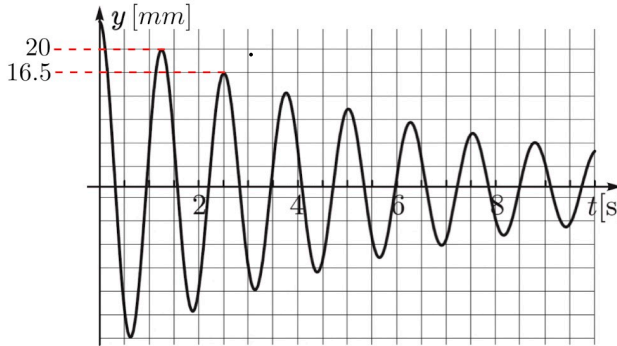


Problem 1: Damped vibrations

Consider a system with a single degree-of-freedom y undergoing a free, underdamped vibration. Let us introduce the *logarithmic decrement* $\Lambda = \ln \frac{y(t)}{y(t+T)}$ with T being the vibration period.

1. What is the correct formula for Lehr's damping ratio D expressed solely in terms of Λ ?
2. Consider the system response shown below. What is the value of D ?



① For an undamped vibration, the frequency is:

$$\omega_d = \sqrt{\omega_0^2 - \delta^2} = \omega_0 \sqrt{1 - D^2} \quad \text{using } D = \frac{\delta}{\omega_0}$$

also the general solution is:

$$y(t) = e^{-\delta t} \sin(\omega_d t + \varphi_0)$$

Now we can plug this into the formula for Λ . We also use that $T = 2\pi/\omega_d$

$$\Lambda = \ln \frac{e^{-\delta t} \sin(\omega_d t + \varphi_0)}{e^{-\delta(t+T)} \sin(\omega_d(t+T) + \varphi_0)}$$

$$= \ln e^{\delta T} = \delta T$$

$$= \underbrace{D \cdot \omega_0}_{\delta} \cdot \frac{2\pi}{\underbrace{\omega_0 \sqrt{1 - D^2}}_T}$$

both sin-terms
are the same
as a shift by
 $T = 2\pi/\omega_d$ has
no effect!

$$\Lambda = \frac{2\pi D}{\sqrt{1-D^2}}$$

now we can solve this for D , which is the final result

$$\Lambda^2 (1-D^2) = 4\pi^2 D^2$$

$$\Lambda^2 = (4\pi^2 + \Lambda^2) D^2$$

$$\Rightarrow \underline{\underline{D = \sqrt{\frac{\Lambda^2}{4\pi^2 + \Lambda^2}}}}$$

② We can find $y(t)$ and $y(t+T)$ from the plot

$$y(t) = 20\text{m}, \quad y(t+T) = 16.5$$

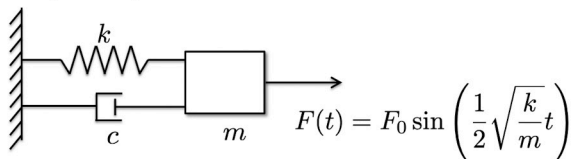
Now we can calculate Λ and with that find D using the result from the first part:

$$\Lambda = \ln \frac{y(t)}{y(t+T)} = 0.19 \Rightarrow \underline{\underline{D = 0.03}}$$

Problem 2: Harmonic oscillator

Consider the harmonic oscillator shown below, which consists of a particle of mass m , an elastic spring of stiffness k , and a dashpot with constant c . The particle is excited by an applied force $F(t) = F_0 \sin\left(\frac{1}{2}\sqrt{\frac{k}{m}}t\right)$.

Given: $m, c, k, F(t) = F_0 \sin\left(\frac{1}{2}\sqrt{\frac{k}{m}}t\right), F_0 > 0$



For what values of c is the amplitude of the steady-state response larger than the static response, i.e., $|\hat{x}| > \frac{F_0}{k}$?

From the lecture we know the equation of motion:

$$m\ddot{x} + c\dot{x} + kx = F(t) = F_0 \sin\left(\frac{1}{2}\sqrt{\frac{k}{m}}t\right)$$

The steady-state response is a vibration at the same frequency as the excitation $F(t)$ with amplitude

$$\hat{x} = V \frac{F_0}{\omega_0^2} = V \frac{F_0}{k}$$

The magnification factor using $\eta = \frac{\Omega}{\omega_0} = \frac{1}{2}$

$$V = \frac{1}{\sqrt{(1-\eta^2)^2 + (2D\eta)^2}} = \frac{1}{\sqrt{\left(1-\frac{1}{4}\right)^2 + \left(2D\frac{1}{2}\right)^2}} = \frac{1}{\sqrt{\frac{9}{16} + D^2}}$$

For $\hat{x} = V \frac{F_0}{k}$ to be larger than $\frac{F_0}{k}$ we see that $V > 1$, so:

$$V > 1 \Rightarrow V^2 > 1 \Rightarrow \frac{9}{16} + D^2 < 1 \Rightarrow D < \sqrt{1 - \frac{9}{16}} = \frac{\sqrt{7}}{4}$$

to now find a condition for the physical damping c we can

express D through c using:

$$D = \frac{s}{\omega_0} = \frac{c}{2m} \cdot \sqrt{\frac{m}{k}} = \frac{c}{2\sqrt{mk}}$$

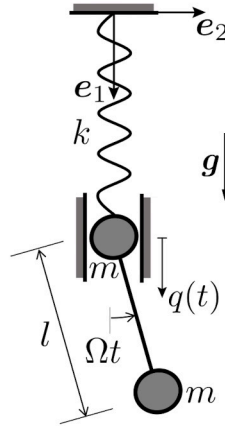
Now we can find our condition for c :

$$\frac{c}{2\sqrt{mk}} < \frac{\sqrt{7}}{4} \Rightarrow \underline{\underline{c < \frac{\sqrt{7}}{2} \sqrt{mk}}}$$

Problem 3: Oscillator with an attached rotator

The shown system consists of a particle of mass m , suspended by a vertical elastic spring of stiffness k and zero unstretched length. The particle is in a frictionless guide, which forces it to move only vertically. A second particle of the same mass m is attached to the first by an inextensible, massless beam of length l , as indicated below. The beam is driven to rotate counterclockwise about the first mass with a constant angular velocity Ω . Gravity acts downwards, as shown. Let $q(t)$ denote the (small) displacement in the e_1 -direction of the suspended particle around its equilibrium.

Given: m, k, l, g, Ω



1. What is the equation of motion of the system in terms of $q(t)$?
2. Consider the initial conditions $q(0) = 0$, and $\dot{q}(0) = v_0$ with a known constant v_0 . Further assume $\Omega^2 = k/m$. What is the motion $q(t)$?

① To find the equation of motion we can either use LMB or the Lagrange equations.

i) First let us use LMB for the CM of the system:

$$(m+m) \underline{a}_{CM} = \underline{F}^{ext}$$

to find \underline{a}_{CM} , we need \underline{r}_{CM} and take the second derivative:

$$\underline{r}_{CM} = \frac{1}{2m} \left[\underbrace{m x_1 \underline{e}_1}_{1st\ particle} + \underbrace{m(x_1 + l \cos(\Omega t)) \underline{e}_1 + m l \sin(\Omega t) \underline{e}_2}_{2nd\ particle} \right]$$

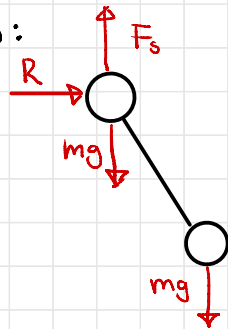
$$\underline{r}_{CM} = \left(x_1 + \frac{l}{2} \cos(\Omega t) \right) \underline{e}_1 + \frac{l}{2} \sin(\Omega t) \underline{e}_2$$

$$\Rightarrow \underline{\dot{r}}_{CM} = \underline{v}_{CM} = \left(\dot{x}_1 - \frac{L}{2} \Omega \sin(\Omega t) \right) \underline{e}_1 + \frac{L}{2} \Omega \cos(\Omega t) \underline{e}_2$$

$$\Rightarrow \underline{\ddot{r}}_{CM} = \underline{a}_{CM} = \left(\ddot{x}_1 - \frac{L}{2} \Omega^2 \cos(\Omega t) \right) \underline{e}_1 - \frac{L}{2} \Omega^2 \sin(\Omega t) \underline{e}_2$$

To determine the external forces we can draw an FBD:

$$\Rightarrow \underline{F}^{ext} = (2mg - kx_1) \underline{e}_1 + R \underline{e}_2$$



Now we can plug both \underline{a}_{CM} and \underline{F}^{ext} into our LMB:

$$\underline{e}_1: 2m\ddot{x}_1 + kx_1 = 2mg + mL\Omega^2 \cos(\Omega t) \quad (*)$$

$$\underline{e}_2: R = -mL\Omega^2 \sin(\Omega t)$$

The \underline{e}_1 -component gives us an equation of motion w.r.t x_1 , but we want it to be w.r.t $q(t) = x_1(t) - x_{eq}$, where x_{eq} is the point of static equilibrium, which is

$$kx_{eq} = 2mg \Rightarrow x_{eq} = \frac{2mg}{k}$$

$$\Rightarrow q = x_1 - \frac{2mg}{k} \quad \text{and} \quad \ddot{q} = \ddot{x}_1$$

This we can plug into (*), which then becomes:

$$\underline{\underline{2m\ddot{q} + kq = mL\Omega^2 \cos(\Omega t)}}$$

ii) We can also do the same using the Lagrange equations:

For the kinetic energy we need the velocity of both particles:

$$\underline{v}_1 = \dot{x}_1 \underline{e}_1, \quad \underline{v}_2 = \dot{x}_1 \underline{e}_1 + \underbrace{\Omega L (-\sin(\Omega t) \underline{e}_1 + \cos(\Omega t) \underline{e}_2)}_{\text{velocity transfer } \ddagger}$$

Thus the kinetic energy is:

$$\begin{aligned} T &= \frac{1}{2} m \left[\dot{x}_1^2 + (\dot{x}_1 - \Omega L \sin(\Omega t))^2 + (\Omega L \cos(\Omega t))^2 \right] \\ &= \frac{1}{2} m \left[\dot{x}_1^2 + \dot{x}_1^2 - 2x_1 \Omega L \sin(\Omega t) + \underbrace{\Omega^2 L^2 \sin^2(\Omega t)}_{\text{---}} \right. \\ &\quad \left. + \underbrace{\Omega^2 L^2 \cos^2(\Omega t)}_{\text{---}} \right] \end{aligned}$$

The potential energy of the system is:

$$V = \frac{1}{2} k x_1^2 - 2mgx_1 - l \cos(\Omega t) \cdot m \cdot g$$

↓
these terms drop out
in the Lagrange eqn.

Using the Lagrangian $L = T - V$ the Lagrange equations are:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_1} \right) - \frac{\partial L}{\partial x_1} = 0 \quad (\text{no non-conservative forces})$$

Using that T only depends on \dot{x}_1 and V only on x_1 we can write this as:

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_1} \right) + \frac{\partial V}{\partial x_1} = 0$$

Now we can plug in T and V and find (in T lots of terms drop out!):

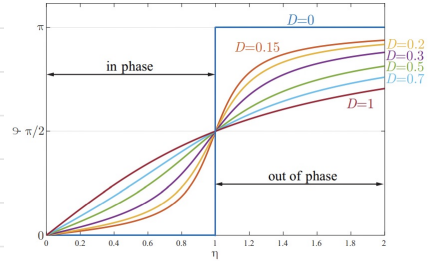
$$\Rightarrow 2m\ddot{x}_1 - m\Omega^2 L \cos(\Omega t) + kx_1 - 2mg = 0$$

This the same as (*) which we found with LMB. Once again we can express this w.r.t. $q = x_1 - x_{eq}$ and find:

$$\underline{\underline{2m\ddot{q} + kq = mL\Omega^2 \cos(\Omega t)}}$$

② We observe that the system is undamped. From the equation of motion we know $\omega_0 = \sqrt{k/2m}$. We are also given $\Omega = \sqrt{k/m} = \sqrt{2}\omega_0$

We see that $\Omega > \omega_0$. As the system is undamped we see from the phase diagram that $\varphi = \pi$



Let us first find the particular solution, which is given in the formula collection as

$$q_p(t) = A \cos(\Omega t - \pi) \Rightarrow \ddot{q}_p(t) = -\Omega^2 A \cos(\Omega t - \pi)$$

We can substitute this into the equation of motion:

$$-2m\Omega^2 A \cos(\Omega t - \pi) + k A \cos(\Omega t - \pi) = mL\Omega^2 \cos(\Omega t)$$

We can use that $\cos(\Omega t - \pi) = -\cos(\Omega t)$ to solve this for A

$$\Rightarrow A = \frac{mL\Omega^2}{2m\Omega^2 - k}, \text{ using } \Omega = \sqrt{k/m} \Rightarrow A = L$$

Next we also need our homogenous solution, which for an undamped system is generally given as:

$$q_h(t) = A_1 \sin(\omega_0 t) + A_2 \cos(\omega_0 t), \quad \omega_0 = \sqrt{\frac{k}{2m}}$$

Thus the complete solution is:

$$q(t) = A_1 \sin(\omega_0 t) + A_2 \cos(\omega_0 t) - L \cos(\sqrt{2}\omega_0 t)$$

Now we can use the initial conditions to find A_1 and A_2

$$\cdot q(0) = A_2 - L = 0 \Rightarrow A_2 = L$$

$$\cdot \dot{q}(0) = A_1 \omega_0 = v_0 \Rightarrow A_1 = \frac{v_0}{\omega_0}$$

We can plug this in and find the final solution:

$$\underline{\underline{q(t) = L \left(\cos(\omega_0 t) - \cos(\sqrt{2} \omega_0 t) \right) + \frac{v_0}{\omega_0} \sin(\omega_0 t)}}$$