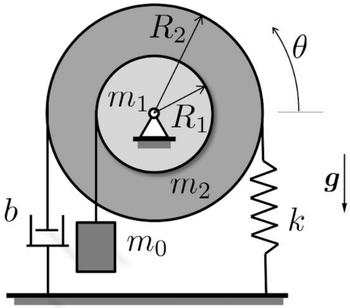
Problem 1: Composite cylinder vibration

Two uniform cylinders of masses m_1 and m_2 and radii R_1 and R_2 are welded together, as shown below. This composite cylinder rotates without friction about a hinge located at its center. An inextensible massless string is wrapped without slipping around the larger cylinder, whose two ends are connected to the ground through a spring of stiffness k and a damper of damping coefficient b. The smaller cylinder is connected to a weight m_0 via another inextensible massless string wrapped without slipping around the smaller cylinder. The weight is constrained to move vertically. Assume that the strings are able to sustain compressive forces, and let θ be a small angle. The spring is unstretched when $\theta = 0$.

Given: m_0, m_1, m_2, R_1, R_2, k, b, $|\theta(t)| \ll 1$, g



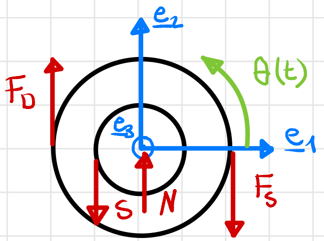
1. What is the equation of motion of the system in terms of the rotation angle $\theta(t)$?
2. At which rotation angle θ_{eq} is the system in static equilibrium?
3. What is the natural frequency ω_0 and characteristic damping δ of the system?

① To solve this problem we'll use the Lagrange equations. For this it's a good idea to start with an FBD.

In this case the force caused by the dashpot and spring are:

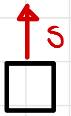
$$F_D = b R_2 \dot{\theta}$$

$$F_s = k R_2 \theta$$



The distance travelled by the weight is $R_1 \theta$

Now we need the kinetic energy of the entire system:



$$T = \frac{1}{2}\left(\frac{1}{2} m_1 R_1^2 \dot{\theta}^2 + \frac{1}{2} m_2 R_2^2 \dot{\theta}^2 + m_0 R_1^2 \dot{\theta}^2\right)$$

and the potential energy of the spring and gravity:

$$V = \frac{1}{2} k R_2^2 \theta^2 - m_0 g R_1 \theta$$

The dashpot causes a non-conservative force, from which we get the generalized force:

$$Q_{nc} = \underline{F}_d \cdot \frac{\partial \underline{r}}{\partial \theta} \quad , \text{ where } \quad \underline{F}_d = b R_2 \dot{\theta} \, \underline{e}_2$$

$\underline{r}$ is the point of application of the force. We need an expression for $\underline{r}$ w.r.t to θ, which we can find using trigonometry

$$\underline{r} = R_2 \cos(\theta + \pi) \, \underline{e}_1 + R_2 \sin(\theta + \pi) \, \underline{e}_2$$

where we use $\theta + \pi$, as for $\theta = 0$, the force applies at $-R \underline{e}_1$! Now we can use our small angle assumption, do a Taylor expansion to get $\cos(\theta + \pi) \approx -1$ and $\sin(\theta + \pi) \approx -\theta$, which allows us to write

$$\Rightarrow \underline{r} \approx -R_2 \underline{e}_1 - R_2 \theta \, \underline{e}_2$$

$$\Rightarrow \frac{\partial \underline{r}}{\partial \theta} \approx -R_2 \underline{e}_2$$

With this we can now find our generalized force for the dashpot:

$$Q_{nc} = \underline{F}_d \cdot \frac{\partial \underline{r}}{\partial \theta} = -b R_2^2 \dot{\theta}$$

Now we have everything we need for our Lagrange equations:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial L} = -b R_2^2 \dot{\theta} \quad , \text{with } L = T - V$$

which after insertion gives the following equation of motion:

$$\left[m_0 R_1^2 + \frac{m_1}{2} R_1^2 + \frac{m_2}{2} R_2^2 \right] \ddot{\theta} + b R_2^2 \dot{\theta} + k R_2^2 \theta = m_0 g R_1$$

② To find a static equilibrium we simply have to solve

$$\left.\frac{\partial V}{\partial \theta}\right|_{\theta_{eq}} = 0$$

$$\Rightarrow kR_2^2\theta_{eq} - m_0 g R_1 = 0$$

$$\Rightarrow \underline{\underline{\theta_{eq} = \frac{m_0 g R_1}{kR_2^2}}}$$

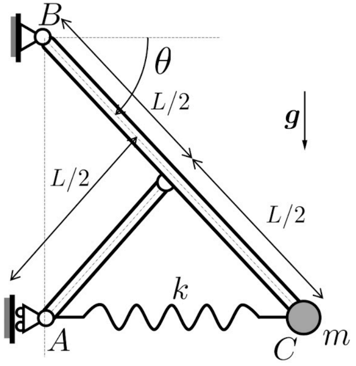
③ The natural frequency and characteristic can be found using the coefficients of the equation of motion.

$$\omega_0 = \sqrt{\frac{kR_2^2}{m_0 R_1^2 + \frac{1}{2}m_1 R_1^2 + \frac{1}{2}m_2 R_2^2}} \quad , \quad \delta = \frac{bR_2^2}{2\cdot\left(m_0 R_1^2 + \frac{1}{2}m_1 R_1^2 + \frac{1}{2}m_2 R_2^2\right)}$$

Problem 2: Bar-spring system I

Consider a system that consists of two bars of negligible masses and of lengths L and $L/2$. The bars connect three points A, B and C, as sketched below. A spring of stiffness k and unstretched length L_0 connects A and C. Furthermore, a particle of mass m is rigidly connected to point C. The system is hinged at B, while point A is constrained to move only in the vertical direction. Gravity acts downwards, as shown.

Given: m, L, k, g



1. For what value of L_0 is $\theta = \pi/4$ an equilibrium position?

For the next task denote the unstretched spring length simply by L_0.

2. For which values of k is $\theta = \pi/4$ a *stable* equilibrium position?

① For $\pi/4$ to be an equilibrium $\left.\frac{\partial V}{\partial \theta}\right|_{\theta=\frac{\pi}{4}} = 0$ needs to be fulfilled.

We can calculate the total potential energy as

$$V = -mgL\sin(\theta) + \frac{1}{2}k\left(L\cos(\theta) - L_0\right)^2$$

$$\Rightarrow \frac{\partial V}{\partial \theta} = -mgL\cos(\theta) - k\left(L\cos(\theta) - L_0\right)L\sin(\theta)$$

$$\left.\frac{\partial V}{\partial \theta}\right|_{\pi/4} = -mgL\frac{\sqrt{2}}{2} - k\left(L\frac{\sqrt{2}}{2} - L_0\right)\frac{\sqrt{2}}{2} \overset{!}{=} 0$$

$$\Rightarrow \underline{\underline{L_0 = \frac{mg}{k} + \frac{\sqrt{2}}{2}L}}$$

② A stable equilibrium requires $\left.\frac{\partial^2 V}{\partial \theta^2}\right|_{\theta=\pi/4} > 0$ to be fulfilled.

$$\frac{\partial^2 V}{\partial \theta^2} = mgL\sin(\theta) + kL^2\sin^2(\theta) - kL^2\cos^2(\theta) - kLL_0\cos(\theta)$$

$$\left.\frac{\partial^2 V}{\partial \theta^2}\right|_{\pi/4} = mgL\frac{\sqrt{2}}{2} + kL^2\frac{1}{2} - kL^2\frac{1}{2} - kLL_0\frac{\sqrt{2}}{2} \overset{!}{>} 0$$

$$\Rightarrow \quad mg + kL_0 > 0$$

which is always satisfied for any $k > 0$ (which is generally the case).